CONTINUOUS DEPENDENCE ON THE COEFFICIENTS AND GLOBAL EXISTENCE FOR STOCHASTIC REACTION DIFFUSION EQUATIONS

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Abstract. We prove convergence of the solutions X_n of semilinear stochastic evolution equations on a Banach space B, driven by a cylindrical Brownian motion in a Hilbert space H,

$$\begin{aligned} dX_n(t) &= (A_n X(t) + F_n(t, X_n(t)))\,dt + G_n(t, X_n(t))\,dW_H(t),\\ X_n(0) &= \xi_n, \end{aligned}$$

assuming that the operators A_n converge to A and the locally Lipschitz functions F_n and G_n converge to the locally Lipschitz functions F and G in an appropriate sense. Moreover, we obtain estimates for the lifetime of the solution X of the limiting problem in terms of the lifetimes of the approximating solutions X_n.

We apply the results to prove global existence for reaction diffusion equations with multiplicative noise and a polynomially bounded reaction term satisfying suitable dissipativity conditions. The operator governing the linear part of the equation can be an arbitrary uniformly elliptic second order elliptic operator.

1. Introduction

The aim of this paper is to address the problem of continuous dependence upon the 'data' A, F, G, and ξ, of the solutions of semilinear stochastic evolution equations of the form

$$\text{(SCP)}\qquad \begin{aligned} dX(t) &= (AX(t) + F(t, X(t)))\,dt + G(t, X(t))\,dW_H(t),\\ X(0) &= \xi, \end{aligned}$$

where A is an unbounded linear operator on a Banach space E, W_H is a cylindrical Brownian motion in a Hilbert space H, and F and G are locally Lipschitz continuous coefficients. This continues a line of research initiated in [?] where the case of globally Lipschitz continuous coefficients was considered. Convergence of solutions in the locally Lipschitz case considered in the present article was posed as an open problem in [?].

In order to outline our approach, we start by briefly recalling how a solution $X = \mathrm{sol}(A, F, G, \xi)$ of equation (SCP) may be found in the case of locally Lipschitz continuous coefficients (see [?, ?, ?]).

For each $r > 0$ one picks functions $F^{(r)}$ and $G^{(r)}$ which are globally Lipschitz continuous and of linear growth and which coincide with F and G on the ball $B^{(r)} = \{x \in E : \|x\| \leqslant r\}$. Then, denoting by $X^{(r)}$ the solution of (SCP) with F and G replaced with $F^{(r)}$ and $G^{(r)}$ respectively, one proves that with

$$\tau^{(r)} := \inf\{t > 0 : \|X^{(r)}(t)\| > r\}$$

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one has $X^{(r)} \equiv X^{(s)}$ on $[0, \tau^{(r)}]$ for all $0 < s \leqslant r$. In particular, $\tau^{(r)}$ increases with r. One then defines $\sigma := \lim_{r\to\infty} \tau^{(r)}$ and, for $t \in [0, \tau^{(r)}]$, $X(t) = X^{(r)}(t)$. It is then shown that X is the maximal solution of the original problem (SCP). The stopping time σ is called the *lifetime* of X.

Suppose now that we approximate the operator A by a sequence of operators A_n, the coefficients F and G by a sequence of coefficients F_n and G_n, and the initial value ξ by a sequence ξ_n. For each $r > 0$ this gives rise to processes $X_n^{(r)}$ from which the solution $X_n = \mathrm{sol}(A_n, F_n, G_n, \xi_n)$ with lifetime σ_n is constructed as above. By the above, one expects convergence $X_n^{(r)} \to X^{(r)}$ as $n \to \infty$ for each $r > 0$, and hence $X_n \to X$ as $n \to \infty$ up to suitable stopping times. The aim of this paper is to describe a general procedure which allows one to deduce that, in these circumstances, one indeed obtains convergence $X_n \to X$, and the lifetime σ of X can be computed explicitly in terms of the lifetimes σ_n of X_n in terms of the stopping times

$$\rho_n^{(r)} := \inf\{t \in (0, \sigma_n) : \|X_n(t)\| > r\}.$$

This follows from a general convergence result for processes defined up to stopping times presented in Section 2.

Applications to stochastic evolution equations are presented in Section 3. In particular, we are able to identify situations in which the limiting process X is globally defined when the processes X_n have this property.

An example where this happens arises in the theory of stochastic reaction diffusion equations. In Section 4 we prove global existence for such equations assuming that the nonlinearity F is of polynomial growth and satisfies suitable dissipativity assumptions and that G is locally Lipschitz and of linear growth. This improves previous results due to Brzeźniak and Gątarek [?] and Cerrai [?] in various ways. Indeed, in our framework, the operator A governing the linear part of the equations can be an arbitrary uniformly elliptic second-order operator. For such operators A, martingale solutions were obtained in [?] for polynomially bounded F and uniformly bounded G. Assuming rather restrictive simultaneous diagonisability conditions on A and the driving noise, in [?] global mild solutions were obtained for polynomially bounded F and certain unbounded nonlinearities G.

In Section 3 and 4 we extend these results by proving global existence of mild solutions under the same growth assumptions on F and G as in [?] but without any diagonisability assumptions on A and the noise process whatsoever. Although our approach combines certain essential features of [?] with a Gronwall type lemma in the spirit of [?], the the abstract results of Section 2 streamline the proof considerably.

In the final section 5 we apply out our results to stochastic reaction diffusion equations driven by white noise in dimension $d = 1$ and driven by a Banach space valued Brownian motion in for the dimension $d \geqslant 2$. Note that the results of [?] do not cover dimensions $d \geqslant 2$ for the Laplace operator on the domain $\mathscr{O} = \{|x| < 1\}$ in dimensions $d \geqslant 2$, as the hypothesis (H1) made in the paper is not satisfied for this operator. Another improvement is that we obtain solutions with trajectories in $C([0, T]; C(\overline{\mathscr{O}}))$ rather than in $C((0, T]; C(\overline{\mathscr{O}})) \cap L^\infty([0, T]; C(\overline{\mathscr{O}}))$.

Notations and terminology are standard and follow those of [?]. Throughout this article we fix probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$, where $0 < T < \infty$ is a finite time horizon. Unless stated otherwise, all processes considered are defined on this probability space, and adaptedness is understood relative to $\mathbb{F}$. We work over the real scalar field, but occasional sectoriality arguments require passage to complexifications; this will be done without further notice.

2. Convergence of locally defined processes

We begin by proving a general convergence result for sequences of processes defined up to certain stopping times. For each $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, a continuous adapted process $X_n = (X_n(t))_{t\in[0,\sigma_n)}$ with values in a Banach space E is given. Here, $\sigma_n : \Omega \to (0,T]$ denotes the explosion time of X_n, i.e., on the set $\{\sigma_n < T\}$ we have $\limsup_{t\uparrow\sigma_n} \|X_n(t)\| = \infty$. For each $r > 0$ and $n \in \overline{\mathbb{N}}$ we set

$$\rho_n^{(r)} := \inf\big\{t \in (0,\sigma_n) : \|X_n(t)\| > r\big\}$$

with the convention $\inf(\varnothing) = T$. Furthermore, we assume that for each $r > 0$ we are given a globally defined, continuous, adapted process $X_n^{(r)} = (X_n^{(r)}(t))_{t\in[0,T]}$ such that the following conditions are satisfied:

(a) For all $n \in \overline{\mathbb{N}}$ and $r > 0$, almost surely

$$X_n^{(r)} \mathbb{1}_{[0,\rho_n^{(r)}]} = X_n \mathbb{1}_{[0,\rho_n^{(r)}]} \quad \text{on } [0,T],$$

(b) For all $r > 0$,

$$\lim_{n\to\infty} X_n^{(r)} = X_\infty^{(r)} \quad \text{in } L^0(\Omega; C([0,T];E)).$$

Here, for a Banach space F, we denote by $L^0(\Omega;F)$ the linear vector space of strongly measurable functions from Ω to F, identifying functions that are equal almost surely. The topology of convergence in probability on $L^0(\Omega;F)$ is metrisable by putting $d(f,g) = \mathbb{E}(\|f-g\| \wedge \mathbb{1})$. This metric turns $L^0(\Omega;F)$ into a complete metric space.

In (a), on the set $\{\rho_n^{(r)} = 0\}$ we do require $X_n^{(r)}(0) = X_n(0)$ almost surely. In the applications below, the processes X_n are obtained by solving certain stochastic evolution equations with locally Lipschitz continuous coefficients, and the processes $X_n^{(r)}$ are obtained as the solutions of the equations with the same initial condition but with coefficients 'frozen' outside the ball of radius r.

We denote by $B_{\rm b}([0,T];E)$ the Banach space of all bounded, strongly Borel measurable functions from $[0,T]$ to E.

Theorem 2.1. *Under the above assumptions, the following assertions hold.*

(1) *For all $r > 0$ and $\varepsilon > 0$ we have, almost surely,*

$$\liminf_{n\to\infty} \rho_n^{(r)} \leqslant \rho_\infty^{(r)} \leqslant \limsup_{n\to\infty} \rho_n^{(r+\varepsilon)}.$$

Moreover, along every subsequence n_k we can find a further subsequence n_{k_j} for which we have, almost surely,

$$\limsup_{j\to\infty} \rho_{n_{k_j}}^{(r)} \leqslant \rho_\infty^{(r)} \leqslant \liminf_{j\to\infty} \rho_{n_{k_j}}^{(r+\varepsilon)}.$$

(2) *For all $r > 0$ and $\varepsilon > 0$ we have*

$$X_n \mathbb{1}_{[0,\rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)}]} \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)}]} \quad \textit{in } L^0(\Omega; B_{\rm b}([0,T];E)).$$

(3) *We have*

$$X_n \mathbb{1}_{[0,\sigma_\infty\wedge\sigma_n)} \to X_\infty \mathbb{1}_{[0,\sigma_\infty)} \quad \textit{in } L^0(\Omega\times[0,T];E).$$

Remark 2.2. Note that the inequalities in (1) involve the whole sequences $(\rho_n^{(r)})_{n\in\mathbb{N}}$ and $(\rho_n^{(r+\varepsilon)})_{n\in\mathbb{N}}$. For this reason we cannot pass to an almost surely uniformly convergent subsequence in (b) and thereby reduce the theorem to a statement about individual trajectories (and hence to a theorem on deterministic functions). Limes inferior and limes superior are highly unstable with respect to passing to a subsequence; for example, the Haar functions h_n on the unit interval satisfy $\liminf_{n\to\infty} h_n = -1$ and $\limsup_{n\to\infty} h_n = 1$, but each subsequence has a further subsequence converging to 0 pointwise almost everywhere.

In the proof of Theorem 2.1 we shall use the following lemma. In its proof and also in the proof of Theorem 2.1 we shall work with versions of X_n and $X_n^{(r)}$ such that (a) holds everywhere on Ω.

Lemma 2.3. *For all $n \in \overline{\mathbb{N}}$, $r > 0$, $\varepsilon > 0$, and $\tau \in (0, T]$ the following holds. If, for some $\omega \in \Omega$, $\|X_n^{(r+\varepsilon)}(t,\omega)\| \leqslant r$ for all $t \in [0,\tau]$, then at least one of the following holds:*

(i) $X_n^{(r+\varepsilon)}(t,\omega) = X_n(t,\omega)$ *for all* $t \in [0,\tau]$*;*
(ii) $\rho_n^{(s)}(\omega) = 0$ *for all* $s \in [0, r+\varepsilon)$.

Proof. We distinguish two cases.

Case 1: If $\rho_n^{(r+\varepsilon)}(\omega) \geqslant \tau$, then $X_n^{(r+\varepsilon)}(t,\omega) = X_n(t,\omega)$ for all $t \in [0,\tau]$ by assumption (a).

Case 2: Suppose that $\rho_n^{(r+\varepsilon)}(\omega) < \tau$ and let $s \in (r, r+\varepsilon)$. Assume that $\rho_n^{(s)}(\omega) > 0$. By path continuity, $0 < \rho_n^{(s)} < \rho_n^{(r+\varepsilon)} < \tau$ and $\|X_n(\rho_n^{(s)}(\omega),\omega)\| = s$. By (a) the contradiction $s = \|X_n(\rho_n^{(s)}(\omega),\omega)\| = \|X_n^{(r+\varepsilon)}(\rho_n^{(s)}(\omega),\omega)\| \leqslant r$ follows. Hence we must have $\rho_n^{(s)} = 0$. Since $\rho_n^{(s)} = 0$ for $s \in (r, r+\varepsilon)$, we obviously have $\rho_n^{(s)} = 0$ for all $s \in [0, r+\varepsilon)$. ☐

Proof of Theorem 2.1. Proof of (1) – We begin with the proof of the left-hand side inequality in first assertion.

Fix $r > 0$. By (b) we may pass to a subsequence which satisfies $X_{n_k}^{(4r)} \to X_\infty^{(4r)}$ in $C([0,T];E)$ almost surely, say for all ω is a set Ω' of full probability. Our first aim is to prove that

$$\limsup_{k\to\infty} \rho_{n_k}^{(r)} \leqslant \rho_\infty^{(r)} \tag{2.1}$$

on Ω'; noting that we could also have started from an arbitrary subsequence, this will also give the left-hand side estimate in the second assertion of (1).

Fix an $\omega \in \Omega'$. We may assume that $\rho_\infty^{(r)}(\omega) < T$, since otherwise (2.1) holds trivially. Likewise we may assume that $\limsup_{k\to\infty} \rho_{n_k}^{(2r)}(\omega) > 0$. For if we had $\limsup_{k\to\infty} \rho_{n_k}^{(2r)}(\omega) = 0$, then certainly $\limsup_{k\to\infty} \rho_{n_k}^{(r)}(\omega) = 0$ and again (2.1) holds trivially.

We claim that in this situation $\rho_\infty^{(2r)}(\omega) > 0$. To prove the claim, observe that since we have $\limsup_{k\to\infty} \rho_{n_k}^{(2r)}(\omega) > 0$, there is a $\delta = \delta(\omega) > 0$ so that, passing to a further subsequence $\rho_{n_{k_j}}^{(2r)} = \rho_{n_{k_{j(\omega)}}}^{(2r)}$ possibly depending on ω, we have $\rho_{n_{k_j}}^{(2r)}(\omega) \geqslant \delta$ for all j. It follows from (a) that $X_{n_{k_j}}(\omega) = X_{n_{k_j}}^{(4r)}(\omega)$ on $[0,\delta]$. Moreover, $X_{n_{k_j}}^{(4r)}$ converges to $X_\infty^{(4r)}(\omega)$, uniformly on $[0,\delta]$. Hence also $X_{n_{k_j}}(\omega)$ converges to $X_\infty^{(4r)}(\omega)$, uniformly on $[0,\delta]$. Now, since $\|X_{n_{k_j}}(t,\omega)\| \leqslant 2r$ for $t \in [0,\delta]$, it follows that $\|X_{n_{k_j}}^{(4r)}(t,\omega)\| \leqslant 2r$ for $t \in [0,\delta]$ which, by Lemma 2.3, implies that $X_{n_{k_j}}(t,\omega) = X_{n_{k_j}}^{(4r)}(t,\omega)$ for such t. By passing to the limit $j \to \infty$ we find $\|X_\infty(t,\omega)\| \leqslant 2r$ for $t \in [0,\delta]$ and thus $\rho_\infty^{(2r)}(\omega) \geqslant \delta > 0$. This proves the claim.

We can now choose a sequence $t_j(\omega) \downarrow \rho_\infty^{(r)}(\omega)$ such that $t_1(\omega) < \rho_\infty^{(2r)}(\omega)$ and $\|X_\infty(t_j(\omega),\omega)\| > r$ for all j. Such a sequence exists by our assumption that $\rho_\infty^{(r)}(\omega) < T$, the definition of $\rho_\infty^{(r)}(\omega)$, and path continuity. For each j there is an index $k_0(\omega, j)$ such that

$$\|X_{n_k}^{(4r)}(\omega) - X_\infty^{(4r)}(\omega)\|_{C([0,T];E)} < \min\big\{\|X_\infty(t_j(\omega),\omega)\| - r, r\big\}$$

for all $k \geqslant k_0(\omega, j)$. For such k we have

$$\|X_{n_k}^{(4r)}(t,\omega)\| < 3r \quad \text{for all } 0 \leqslant t \leqslant \rho_\infty^{(2r)}(\omega).$$

To see this, note that if $0 \leqslant t \leqslant \rho_\infty^{(2r)}(\omega)$, then $\|X_\infty^{(4r)}(t,\omega)\| = \|X_\infty(t,\omega)\| \leqslant 2r$. Also, for all such k we have

$$\|X_{n_k}^{(4r)}(t_j(\omega),\omega)\| > r.$$

By Lemma 2.3, either $\|X_{n_k}(t_j(\omega),\omega)\| > r$ or $\rho_{n_k}^{(r)}(\omega) = 0$. Note that in both cases,

$$\rho_{n_k}^{(r)}(\omega) \leqslant t_j(\omega).$$

This being true for all $k \geqslant k_0(\omega, j)$, it follows that $\limsup_{k\to\infty} \rho_{n_k}^{(r)}(\omega) \leqslant t_j(\omega)$. Taking the infimum over j, we see that $\limsup_{k\to\infty} \rho_{n_k}^{(r)}(\omega) \leqslant \rho_\infty^{(r)}(\omega)$. This proves (2.1).

Now fix $\eta > 0$. On the set $\bigcup_{m\in\mathbb{N}} \bigcap_{n\geqslant m} \{\rho_n^{(r)} \geqslant \rho_\infty^{(r)} + \eta\}$, the above subsequence certainly satisfies $\limsup_{k\to\infty} \rho_{n_k}^{(r)} \geqslant \rho_\infty^{(r)} + \eta$. But since (2.1) holds on a set of full probability, this implies that $\mathbb{P}(\bigcup_{m\in\mathbb{N}} \bigcap_{n\geqslant m} \{\rho_n^{(r)} \geqslant \rho_\infty^{(r)} + \eta\}) = 0$. It follows that

$$\mathbb{P}\big(\liminf_{n\to\infty} \rho_n^{(r)} \leqslant \rho_\infty^{(r)} + \eta\big) \geqslant \mathbb{P}\big(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geqslant m} \{\rho_n^{(r)} \leqslant \rho_\infty^{(r)} + \eta\}\big) = 1\,.$$

Upon letting $\eta \downarrow 0$ we have $\{\liminf_{n\to\infty} \rho_n^{(r)} \leqslant \rho_\infty^{(r)} + \eta\} \downarrow \{\liminf_{n\to\infty} \rho_n^{(r)} \leqslant \rho_\infty^{(r)}\}$, from which it follows that $\mathbb{P}(\liminf_{n\to\infty} \rho_n^{(r)} \leqslant \rho_\infty^{(r)}) = 1$.

Next we prove the right-hand side inequality of the first assertion in (1).

Fix $r > 0$ and $\varepsilon > 0$. By (b) we may pass to a subsequence such that $X_{n_k}^{(r+2\varepsilon)} \to X_\infty^{(r+2\varepsilon)}$ in $C([0,T];E)$ almost surely, say on the set Ω' of full probability. Our first aim is to prove that

$$\liminf_{k\to\infty} \rho_{n_k}^{(r+\varepsilon)} \geqslant \rho_\infty^{(r)} \tag{2.2}$$

on Ω'; noting that we could also have started from an arbitrary subsequence, this will also give the right-hand side estimate in the second assertion of (1).

Fix an $\omega \in \Omega'$. We may assume that $\rho_\infty^{(r)}(\omega) > 0$, for otherwise (2.2) trivially holds.

The next step is to prove that $X_{n_k}(\omega) \to X_\infty(\omega)$ uniformly on $[0, \rho_\infty^{(r)}(\omega)]$. On this interval we know that $\|X_\infty(\omega)\| \leqslant r$. Hence, by (a), $X_\infty(\omega) = X_\infty^{(r+2\varepsilon)}(\omega)$ on $[0, \rho_\infty^{(r)}(\omega)]$. Since $X_{n_k}^{(r+2\varepsilon)}(\omega) \to X_\infty^{(r+2\varepsilon)}(\omega)$ uniformly, it follows that, for large enough k, say for all $k \geqslant k_1(\omega)$,

$$\|X_{n_k}^{(r+2\varepsilon)}(\omega)\| \leqslant r + \varepsilon \quad \text{on} \quad [0, \rho_\infty^{(r)}(\omega)]. \tag{2.3}$$

By (2.3) and Lemma 2.3, for each $k \geqslant k_1(\omega)$ we are in at least one of the following two cases: either we have $\|X_{n_k}(\omega)\| \leqslant r + \varepsilon$ on $[0, \rho_\infty^{(r)}(\omega)]$ and thus $\rho_{n_k}^{(r+\varepsilon)}(\omega) \geqslant \rho_\infty^{(r)}(\omega)$, or else $\rho_{n_k}^{(r+\varepsilon)}(\omega) = 0$.

Suppose the latter happens for infinitely many k (the set of these k may depend on ω). Then $\|X_{n_k}(0,\omega)\| \geqslant r + \varepsilon$ for infinitely many k. Since

$$X_{n_k}(0,\omega) = X_{n_k}^{(r+2\varepsilon)}(0,\omega) \to X_\infty^{(r+2\varepsilon)}(0,\omega) = X_\infty(0,\omega)$$

this implies $\|X_\infty(0,\omega)\| \geqslant r + \varepsilon$. But then $\rho_\infty^{(r)}(\omega) = 0$ by path continuity, and this contradicts our previous assumption. Thus, for all but finitely many k we must have the first alternative. This proves (2.2).

Fix $\eta > 0$. Arguing as above, $\mathbb{P}(\bigcup_{m\in\mathbb{N}} \bigcap_{n\geqslant m} \{\rho_n^{(r+\varepsilon)} \leqslant \rho_\infty^{(r)} - \eta\}) = 0$ and thus $\mathbb{P}(\limsup_{n\to\infty} \rho_n^{(r+\varepsilon)} \geqslant \rho_\infty^{(r)} - \eta) = 1$. Upon letting $\eta \downarrow 0$ we see that $\mathbb{P}(\limsup_{n\to\infty} \rho_n^{(r+\varepsilon)} \geqslant \rho_\infty^{(r)}) = 1$.

Proof of (2) – Fix $r > 0$ and $\varepsilon > 0$. Since convergence in probability is metrisable, it suffices to prove that every subsequence of $(X_n \mathbb{1}_{[0,\rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)}]})_{n\in\mathbb{N}}$ has a further subsequence for which the claimed convergence holds.

Given a subsequence, we may pass to a further subsequence (which, for ease of notation, we index by n again) such that

$$X_n^{(r)} \to X_\infty^{(r)} \text{ and } X_n^{(r+2\varepsilon)} \to X_\infty^{(r+2\varepsilon)} \quad \text{in } C([0,T];E) \text{ almost surely.} \tag{2.4}$$

Fix an ω from the set of convergence. If $\rho_\infty^{(r)}(\omega) = 0$, then it follows from the first assumption in (2.4) that $X_n \mathbb{1}_{[0,\rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)}]}(\omega) \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)}]}(\omega)$. Therefore in the rest of the argument we may assume that $\rho_\infty^{(r)}(\omega) > 0$. Then, as we have seen in the proof of the second assertion of (1), for all $n \geqslant n_0(\omega)$ we have $\|X_n(t,\omega)\| \leqslant r + \varepsilon$ for all $0 \leqslant t \leqslant \rho_\infty^{(r)}(\omega)$. For these n we see that $\rho_n^{(r+\varepsilon)}(\omega) \geqslant \rho_\infty^{(r)}(\omega)$ and therefore $X_n \mathbb{1}_{[0,\rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)}]}(\omega) = X_n \mathbb{1}_{[0,\rho_\infty^{(r)}]}(\omega)$. Also, $X_n(t,\omega) = X_n^{(r+2\varepsilon)}(t,\omega)$ and $X_\infty(t,\omega) = X_\infty^{(r+2\varepsilon)}(t,\omega)$ for $0 \leqslant t \leqslant \rho_\infty^{(r)}(\omega) \wedge \rho_n^{(r+\varepsilon)}(\omega)$. Combining these observations with (2.4) we find, for $n \geqslant n_0(\omega)$,

$$\begin{aligned} X_n \mathbb{1}_{[0,\rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)}]}(\omega) = X_n \mathbb{1}_{[0,\rho_\infty^{(r)}]}(\omega) &= X_n^{(r+2\varepsilon)} \mathbb{1}_{[0,\rho_\infty^{(r)}]}(\omega) \\ &\to X_\infty^{(r+2\varepsilon)} \mathbb{1}_{[0,\rho_\infty^{(r)}]}(\omega) = X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)}]}(\omega) \end{aligned}$$

in $B_{\mathrm{b}}([0,T];E)$.

Proof of (3) – Again, we will show that every subsequence has a subsequence for which the claimed convergence holds.

Let a subsequence be given. By the proof of (2), this subsequence has a further subsequence $n_{k,1}$ such that

$$X_{n_{k,1}} \mathbb{1}_{[0,\rho_\infty^{(1)} \wedge \rho_{n_{k,1}}^{(2)}]}(\omega) \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(1)}]}(\omega)$$

in $B_{\mathrm{b}}([0,T];E)$ as $k \to \infty$, for all ω outside a set null set N_1.

Suppose we have already constructed a subsequence $n_{k,l}$ such that

$$X_{n_{k,l}} \mathbb{1}_{[0,\rho_\infty^{(j)} \wedge \rho_{n_{k,l}}^{(j+1)}]}(\omega) \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(j)}]}(\omega)$$

in $B_{\mathrm{b}}([0,T];E)$ as $k \to \infty$, for all $j \in \{1, \dots, l\}$ and all ω outside a null set N_l. By the proof of (2), we can extract a further subsequence $n_{k,l+1}$ such that

$$X_{n_{k,l}} \mathbb{1}_{[0,\rho_\infty^{(j)} \wedge \rho_{n_{k,l+1}}^{(j+1)}]}(\omega) \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(j)}]}(\omega)$$

in $B_{\mathrm{b}}([0,T];E)$ as $k \to \infty$, for all $j \in \{1, \dots, l, l+1\}$ and all ω outside a null set N_{l+1}. We continue this procedure inductively.

Now put $N := \bigcup_{l \geqslant 1} N_l$. Setting $n_l := n_{l,l}$, it follows that

$$X_{n_l} \mathbb{1}_{[0,\rho_\infty^{(j)} \wedge \rho_{n_l}^{(j+1)}]}(\omega) \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(j)}]}(\omega) \tag{2.5}$$

in $B_{\mathrm{b}}([0,T];E)$ as $l \to \infty$, for all $j \geqslant 1$ and ω outside the null set N.

By the second part of (1), upon replacing N by some larger null set and passing to a further subsequence of n_l if necessary, we may assume that outside N we also have

$$\liminf_{l\to\infty} \rho_{n_l}^{(j+1)}(\omega) \geqslant \rho_\infty^{(j)}(\omega) \quad \text{for all} \quad j \geqslant 1. \tag{2.6}$$

Now let $(t,\omega) \in [0,T] \times (\Omega \setminus N)$. We claim that

$$X_{n_l}(t,\omega) \mathbb{1}_{[0,\sigma_\infty \wedge \sigma_{n_l})}(t,\omega) \to X_\infty(t,\omega) \mathbb{1}_{[0,\sigma_\infty)}(t,\omega)$$

in E as $l \to \infty$.

We distinguish two cases. First, if $t \geqslant \sigma_\infty(\omega)$, then

$$X_{n_l}(t,\omega) \mathbb{1}_{[0,\sigma_\infty \wedge \sigma_{n_l})}(t,\omega) = 0 = X_\infty(t,\omega) \mathbb{1}_{[0,\sigma_\infty)}(t,\omega)$$

for all $l \in \mathbb{N}$ and there is nothing to prove.

Second, suppose that $t < \sigma_\infty(\omega)$. Pick an integer j such that $\|X_\infty(s,\omega)\| < j$ for all $0 \leqslant s \leqslant t$. Then $t < \rho_\infty^{(j)}(\omega)$. By (2.6), for all large enough l we have $t < \rho_{n_l}^{(j+1)}(\omega) \leqslant \sigma_{n_l}(\omega)$. Hence, for all large l,

$$X_{n_l}(t,\omega)\mathbb{1}_{[0,\sigma_\infty\wedge\sigma_{n_l})}(t,\omega) = X_{n_l}(t,\omega) = X_{n_l}(t,\omega)\mathbb{1}_{[0,\rho_\infty^{(j)}\wedge\rho_{n_l}^{(j+1)}]}(t,\omega).$$

By (2.5), the right-hand side converges to

$$X_\infty(t,\omega) = X_\infty(t,\omega)\mathbb{1}_{[0,\rho_\infty^{(j)}]}(t,\omega) = X_\infty(t,\omega)\mathbb{1}_{[0,\sigma_\infty)}(t,\omega).$$

This proves the claim. □

Corollary 2.4. *Under the above assumptions we have*

$$\sigma_\infty \geqslant \lim_{r\to\infty}\liminf_{n\to\infty}\rho_n^{(r)}$$

almost surely. Furthermore, every subsequence n_k has a further subsequence n_{k_j} for which

$$\sigma_\infty = \lim_{r\to\infty}\liminf_{j\to\infty}\rho_{n_{k_j}}^{(r)}$$

almost surely.

Proof. The first assertion follows from the first assertion in Theorem 2.1(1) upon letting $r \to \infty$. To obtain the second assertion, given a subsequence n_k let n_{k_j} be a subsequence for which the second assertion in Theorem 2.1(1) holds. Then $\sigma_\infty \leqslant \lim_{r\to\infty}\liminf_{j\to\infty}\rho_{n_{k_j}}^{(r)}$ almost surely. The reverse inequality follows from the first part of Theorem 2.1(1) applied to this subsequence. □

Corollary 2.5. *Under the above assumptions, assume that $\sigma_n = T$ almost surely for all $n \in \overline{\mathbb{N}}$. Then $X_n \to X_\infty$ in $L^0(\Omega; C([0,T];E))$.*

Proof. It suffices to show that any subsequence has a further subsequence with the asserted property. Fix a subsequence, which, after relabeling, we shall denote by X_n again.

Fix $N \in \mathbb{N}$. Passing to a further subsequence, by (b) we may assume that

$$X_n^{(N+1)} \to X_\infty^{(N+1)} \text{ in } C([0,T];E) \tag{2.7}$$

almost surely. Passing to yet a further subsequence, by Theorem 2.1 (2) we may assume uniform convergence

$$X_n\mathbb{1}_{[0,\rho_\infty^{(N)}\wedge\rho_n^{(N+1)}]} \to X_\infty\mathbb{1}_{[0,\rho_\infty^{(N)}]} \tag{2.8}$$

almost surely. Let Ω_0 be a set of probability one on which both hold and fix $\omega \in \Omega_0$.

If $X_\infty(\rho_\infty^{(N)}(\omega),\omega) \neq 0$, then by (2.8) necessarily there exists an index $n_0(\omega)$ such that $\rho_n^{(N+1)}(\omega) \geqslant \rho_\infty^{(N)}(\omega)$ for all $n \geqslant n_0(\omega)$.

Suppose next that $X_\infty(\rho_\infty^{(N)}(\omega),\omega) = 0$. We claim that also in this case there exists an index $n_0(\omega)$ such that $\rho_n^{(N+1)}(\omega) \geqslant \rho_\infty^{(N)}(\omega)$ for all $n \geqslant n_0(\omega)$. Indeed, if this were wrong, we could pick a subsequence $n_k(\omega) \to \infty$ such that $\rho_{n_k(\omega)}^{(N+1)}(\omega) < \rho_\infty^{(N)}(\omega)$. Since $\|X_{n_k(\omega)}(\rho_{n_k(\omega)}^{(N+1)},\omega)\| = N+1$ by path continuity, we obtain $\|X_{n_k(\omega)}\mathbb{1}_{[0,\rho_\infty^{(N)}(\omega)\wedge\rho_{n_k(\omega)}^{(N+1)}]} - X_\infty(\omega)\mathbb{1}_{[0,\rho_\infty^{(N)}(\omega)]}\|_\infty \geqslant N+1$, contradicting (2.8). This proves the claim.

By what we have proved so far, it follows that on the set $\Omega_N := \Omega_0 \cap \{\rho_\infty^{(N)} = T\}$ we have $\rho_n^{(N+1)} \geqslant \rho_\infty^{(N)}$ eventually, and therefore $\rho_n^{(N+1)} = T$ eventually. Consequently, by (a), for each $\omega \in \Omega_N$ we have

$$X_n\mathbb{1}_{[0,\rho_\infty^{(N)}\wedge\rho_n^{(N+1)}]}(\omega) = X_n(\omega) = X_n^{(N+1)}(\omega) \quad \text{for } n \geqslant n_0(\omega)$$

and, again by (a),

$$X_\infty\mathbb{1}_{[0,\rho_\infty^{(N)}\wedge\rho_n^{(N+1)}]}(\omega) = X_\infty(\omega) = X_\infty^{(N)}(\omega) = X_\infty^{(N+1)}(\omega) \quad \text{for } n \geqslant n_0(\omega),$$

the last equality being a consequence of the fact that for all $t \in [0, \rho_\infty^{(N)}(\omega)] = [0, T]$ we have $\|X_\infty(t,\omega)\| \leqslant N \leqslant N+1$ plus another application of (a).

Thus, by (2.7), $X_n(\omega) \to X_\infty(\omega)$ in $C([0,T];E)$.

Considering a diagonal sequence, we find a subsequence of X_n which converges to X_∞ almost surely in $C([0,T];E))$ on $\bigcup_{N\in\mathbb{N}} \Omega_N$. Since $\sigma_\infty = T$ almost surely, the latter set has full measure. $\square$

Corollary 2.6. *Under the above assumptions, suppose that $\sigma_n = T$ almost surely for all $n \in \mathbb{N}$, and suppose furthermore that for some $p \geqslant 1$ we have*

$$\sup_{n\in\mathbb{N}} \|X_n\|_{L^p(\Omega;C([0,T];E))} < \infty.$$

Then:

1. *Almost surely, $\sigma_\infty = T$;*
2. *We have $X_\infty \in L^p(\Omega; C([0,T];E))$;*
3. *If $p > 1$, then, for all $1 \leqslant q < p$,*

$$X_n \to X_\infty \quad \text{in} \quad L^q(\Omega; C([0,T];E)).$$

Proof. (1) From Theorem 2.1(2) and Fatou's lemma we infer, for $r > 0$ and $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}\|X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)}]}\|^p_{B_{\rm b}([0,T];E)} &\leqslant \liminf_{n\to\infty} \mathbb{E}\|X_n \mathbb{1}_{[0,\rho_\infty^{(r)}\wedge\rho_n^{(r+\varepsilon)}]}\|^p_{B_{\rm b}([0,T];E)} \\ &\leqslant \liminf_{n\to\infty} \mathbb{E}\|X_n\|^p_{C([0,T];E)} \leqslant C,\end{aligned}$$

where $C := \sup_{n\in\mathbb{N}} \|X_n\|_{L^p(\Omega;C([0,T];E))}$. Employing Fatou's lemma a second time, we see that

$$\begin{aligned}\mathbb{E}\|X_\infty \mathbb{1}_{[0,\sigma_\infty)}\|^p_{B_{\rm b}([0,T];E)} &= \mathbb{E} \lim_{r\to\infty} \|X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)}]}\|^p_{B_{\rm b}([0,T];E)} \\ &\leqslant \liminf_{r\to\infty} \mathbb{E}\|X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)}]}\|^p_{B_{\rm b}([0,T];E)} \leqslant C\,.\end{aligned}$$

In particular, we infer that X_∞ is almost surely bounded on $[0,\sigma_\infty)$. Since σ_∞ is an explosion time, this is only possible if $\sigma_\infty = T$.

(2) From what we have proved so far it follows that $X_\infty \in L^p(\Omega; C_{\rm b}([0,T);E))$ and thus $\sup_{t\in[0,T)} \|X_\infty(t,\omega)\| < \infty$ for almost all $\omega \in \Omega$. By continuity of the paths, $\sup_{t\in[0,T)} \|X_\infty(t,\omega)\| = \sup_{t\in[0,T]} \|X_\infty(t,\omega)\|$ almost surely and now $X_\infty \in L^p(\Omega; C([0,T];E))$ follows.

(3) Follows directly from the boundedness of X_n in $L^p(\Omega; C([0,T];E))$ and the convergence $X_n \to X$ in $L^0(\Omega; C([0,T];E))$ which follows from Corollary 2.5. $\square$

3. Application to semilinear stochastic equations

We shall now apply the abstract results of the previous section to prove convergence of approximate solutions of stochastic evolution equations of the form

$$\text{(SCP)} \qquad \begin{cases} dX(t) = [AX(t) + F(t,X(t))]\,dt + G(t,X(t))\,dW(t) \\ X(0) = \xi. \end{cases}$$

The driving noise process W is assumed to be a cylindrical Brownian motion in some Hilbert space H.

Before addressing equation (SCP), let us first review some terminology needed in what follows.

3.1. **γ-Radonifying operators.** Let H be a real Hilbert space and F a real Banach space. Every finite rank operator $T : H \to F$ can be represented in the form

$$T = \sum_{n=1}^{N} h_n \otimes x_n$$

for some integer $N \geqslant 1$, with $(h_n)_{n=1}^N$ orthonormal in H and $(x_n)_{n=1}^N$ some sequence in F (here $h \otimes x$ is the rank one operator mapping $g \in H$ to $[g, h]_H\, x \in E$). With T represented in this form, we define

$$\|T\|_{\gamma(H,F)} := \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n\, x_n \Big\|^2,$$

where $(\gamma_n)_{n=1}^N$ is a sequence of independent standard normal random variables. This norm is independent of the representation of T in the above form. The space $\gamma(H, F)$ is now defined as the completion of the space of finite rank operators $H \times F$ with respect to the norm $\| \cdot \|_{\gamma(H,F)}$. The identity operator on $H \otimes F$ extends to a continuous embedding $\gamma(H, F) \hookrightarrow \mathscr{L}(H, F)$. Thus we may view $\gamma(H, F)$ as a linear subspace of $\mathscr{L}(H, F)$, and the operators belonging to $\gamma(H, F)$ are called the γ-radonifying operators from H to F.

When F is a Hilbert space, $\gamma(H, F)$ consists precisely of all Hilbert-Schmidt operators from H to F, and this identification is isometric. If $(S, \mathscr{S}, \mu)$ is a σ-finite measure space, then for $F = L^p(S, \mu)$ with $1 \leqslant p < \infty$, the space $\gamma(H, L^p(S, \mu))$ is canonically isomorphic to $L^p(S, \mu; H)$. The isomorphism is obtained by viewing a function $f \in L^p(\mu; H)$ as operator T_f from H to $L^p(S, \mu)$ by defining $T_f h := [f(\cdot), h]_H$.

For more information we refer to the survey article [**?**].

3.2. **UMD spaces.** It is a well-established fact that many results from harmonic analysis and stochastic analysis involving some martingale structure extend to the Banach space setting, provided one restricts oneself to the class of UMD spaces.

Let $1 < p < \infty$. A Banach space E is said to be a *UMD$_p$-space* if there exists a constant β such that for all E-valued L^p-martingale difference sequences $(d_n)_{n=1}^N$ we have

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^p \leqslant \beta^p \mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|^p.$$

The least possible constant β in the above inequalities is called the *UMD$_p$-constant* of E, notation $\beta_p(E)$.

Every Hilbert space H is a UMD$_2$-space, with $\beta_2(H) = 1$. For a σ-finite measure space $(S, \mathscr{S}, \mu)$ and $1 < p < \infty$ the space $L^p(S, \mu)$ is a UMD$_p$-space. If X is a UMD$_p$-space, then so is $L^p(S, \mu; X)$.

It is a non-trivial fact that if a Banach space is UMD$_p$ for some $1 < p < \infty$, then it is UMD$_p$ for all $1 < p < \infty$. Thus we may define a Banach space to be *UMD* if it is UMD$_p$ for some (equivalently, all) $1 < p < \infty$. The term 'UMD' is an abbreviation for 'unconditional martingale differences'. For more information we refer to the survey articles [**?**, **?**].

3.3. **Stochastic evolution equations in UMD spaces.** Under the assumptions stated below, existence and uniqueness of maximal solutions for (SCP) in UMD spaces E was proved in [**?**], and convergence of the solutions in the case of globally Lipschitz continuous coefficients was established in [**?**].

Continuing the notations of the previous section we shall write $A = A_\infty$, $F = F_\infty$, $G = G_\infty$ and $\xi = \xi_\infty$ when we thinks of these objects as the limits of sequences of approximating objects A_n, F_n, G_n, ξ_n.

(A1) For $n \in \overline{\mathbb{N}}$, the operators A_n are densely defined, closed and *uniformly sectorial* on E in the sense that there exist numbers $M \geqslant 1$ and $w \in \mathbb{R}$ such that each A_n is sectorial of type (M, w).

(A2) The operators A_n converge to A_∞ *in the strong resolvent sense*:

$$\lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A_\infty)x$$

for some (equivalently, all) $\operatorname{Re}\lambda > w$ and all $x \in E$.

Assumptions (A1) and (A2) coincide with those made in [?]. Assuming (A1), the operator A_n generates a strongly continuous analytic semigroup $S_n = (S_n(t))_{t\geqslant 0}$ and the semigroups $(e^{-wt}S_n(t))_{t\geqslant 0}$ are uniformly bounded, uniformly in n. Therefore, for $w' > w$ the fractional powers $(w' - A_n)^\alpha$ are well defined for all $\alpha \in (0, 1)$. In particular, the fractional domain spaces

$$E_{n,\alpha} := \mathsf{D}((w' - A_n)^\alpha)$$

are Banach spaces with respect to the norm

$$\|x\|_{E_{n,\alpha}} := \|(w' - A_n)^\alpha x\|.$$

Up to equivalent norms, these spaces are independent of the choice of w'. It may happen, however, that these spaces vary with n. This may cause problems, and to avoid these we make the following assumption.

(A3) For all $0 < \alpha < \frac{1}{2}$ we have $E_{n,\alpha} = E_{\infty,\alpha}$ as linear subspace of E. Moreover, there exist constants $c_\alpha > 0$ and $C_\alpha > 0$ such that

$$c_\alpha\|x\|_{E_{\infty,\alpha}} \leqslant \|x\|_{E_{n,\alpha}} \leqslant C_\alpha\|x\|_{E_{\infty,\alpha}} \quad \forall\, x \in E_\alpha\,,\ n \in \mathbb{N}\,.$$

We then set $E_\alpha := E_{\infty,\alpha}$ and $\|\cdot\|_\alpha := \|\cdot\|_{E_{\infty,\alpha}}$. We complete the scale E_α by setting $E_0 := E$ and $\|\cdot\|_0 := \|\cdot\|$.

Remark 3.1. More generally, one could replace (A3) by the assumption that there exists $\alpha_0 \in (0, 1)$ such that $E_{n,\alpha} = E_{\infty,\alpha}$ holds for $0 < \alpha < \alpha_0$; this would require obvious changes in what follows. It seems that the case $\alpha = \frac{1}{2}$ is most important in applications; see the example at the end of Section 5.

It is immediate from assumption (A3) that for each $0 < \alpha < \frac{1}{2}$, the operators $(w' - A_n)^\alpha$ are uniformly bounded in $\mathscr{L}(E_\alpha, E)$ and that the operators $(w' - A_n)^{-\alpha}$ are uniformly bounded in $\mathscr{L}(E, E_\alpha)$.

For $0 < \alpha < \frac{1}{2}$ we define the extrapolation spaces $E_{n,-\alpha}$ as the completion of E under the norms $\|x\|_{E_{n,-\alpha}} := \|(w' - A_n)^{-\alpha}x\|_E$. For fixed n, these spaces are independent of $w' > w$ up to an equivalent norm, and for each fixed $w' > w$ these spaces are independent of n with equivalence constants independent of n. Accordingly, we set $E_{-\alpha} := E_{\infty,-\alpha}$ and $\|\cdot\|_{-\alpha} := \|\cdot\|_{E_{\infty,-\alpha}}$. Then for all $0 \leqslant \alpha, \beta < \frac{1}{2}$, the operators $(w' - A_n)^{\alpha+\beta}$ and $(w' - A_n)^{-(\alpha+\beta)}$ are uniformly bounded in $\mathscr{L}(E_\alpha, E_{-\beta})$ and $\mathscr{L}(E_{-\beta}, E_\alpha)$, respectively.

Concerning the coefficients F_n and G_n, we shall assume that the hypotheses of [?, Section 8] are satisfied, uniformly with respect to n, and with exponents

$$0 \leqslant \theta < \tfrac{1}{2}, \quad 0 \leqslant \kappa_F, \kappa_G < \tfrac{1}{2},$$

and we add the assumptions concerning their convergence of [?]. The restriction $\theta, \kappa_F, \kappa_B < \frac{1}{2}$ is due to assumption (A3) which only asserts us control of the fractional domain spaces/extrapolation spaces in this range. Our precise assumptions are as follows. We refer to [?] for further explanations.

(F1) The maps $F_n : [0, T] \times \Omega \times E_\theta \to E_{-\kappa_F}$ are uniformly locally Lipschitz continuous, i.e., for all $r > 0$ there exists a constant $L_F^{(r)} \geqslant 0$ such that

$$\|F_n(t, \omega, x) - F_n(t, \omega, y)\|_{-\kappa_F} \leqslant L_F^{(r)}\|x - y\|_\theta$$

for all $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in E_\theta$ of norm $\|x\|_\theta, \|y\|_\theta \leqslant r$. Moreover, for all $x \in E_\theta$ the map $(t,\omega) \mapsto F_n(t,\omega,x)$ is strongly measurable and adapted and there exists a constant $C_{F,0}$ such that

$$\|F(t,\omega,0)\|_{E_{-\kappa_F}} \leqslant C_{F,0}.$$

(F2) For all $r > 0$ and almost all $(t,\omega) \in [0,T] \times \Omega$ we have

$$F_n^{(r)}(t,\omega,x) \to F_\infty^{(r)}(t,\omega,x) \ \text{ in } \ E_{-\kappa_F}$$

for all $x \in E_\theta$.

(G1) The maps $G_n : [0,T] \times \Omega \times E_\theta \to \gamma(H, E_{-\kappa_G})$ are uniformly locally γ-Lipschitz continuous, i.e., for all $r > 0$ there exist maps $G_n^{(r)} : [0,T] \times \Omega \times E_\theta \to \gamma(H, E_{-\kappa_G})$ such that

$$G_n^{(r)} = G_n \quad \text{on} \quad [0,T] \times \Omega \times \{x \in E_\theta \,:\, \|x\|_\theta \leqslant r\}.$$

Moreover, there there exist constants $L_G^{(r)}$ such that for all Borel probability measures μ on $[0,T]$, all $\omega \in \Omega$, all $\phi_1, \phi_2 \in L^2([0,T],\mu;E_\theta) \cap \gamma(L^2([0,T],\mu), E_\theta) =: L^2_\gamma([0,T],\mu;E_\theta)$, and all $n \in \overline{\mathbb{N}}$ we have

$$\begin{aligned}\|G_n^{(r)}(\cdot,\omega,\phi_1) - G_n^{(r)}(\cdot,\omega,\phi_2)\|_{\gamma(L^2([0,T],\mu;H),E_{-\kappa_G})} \\ \leqslant L_G^{(r)} \|\phi_1 - \phi_2\|_{L^2_\gamma([0,T],\mu;E_\theta)}.\end{aligned}$$

For all $x \in E_\theta$, $h \in H$, and $n \in \overline{\mathbb{N}}$ there exists a constant $C_{G,0}$ such that for all Borel probability measures μ on $[0,T]$,

$$\|G_n^{(r)}(\cdot,\omega,0)\|_{\gamma(L^2([0,T],\mu;H),E_{-\kappa_G})} \leqslant C_{G,0}.$$

Finally, we assume that for all $n \in \overline{\mathbb{N}}$, $x \in E_\theta$ and $h \in H$ the map $(t,\omega) \mapsto G_n(t,\omega,x)h$ is strongly measurable and adapted. We also assume this measurability and adaptedness of the maps $G_n^{(r)}$.

(G2) For all $r > 0$ and almost all $(t,\omega) \in [0,T] \times \Omega$ we have

$$G_n^{(r)}(\cdot,\omega,x) \to G_\infty^{(r)}(\cdot,\omega,x) \ \text{ in } \ \gamma(L^2(0,T,\mu;H), E_{-\kappa_G})$$

for all $x \in E_\theta$ and all Borel probability measures μ on $[0,T]$.

Examples where these assumptions are satisfied have been presented in [**?**, **?**].

Recall that a Banach space E is said to have *type* $p \in [1,2]$ if there exists a constant $C_p \geqslant 0$ such that for all finite sequences $x_1, \dots, x_N$ in E we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2\Big)^{\frac12} \leqslant C_p\Big(\sum_{n=1}^N \|x_n\|^p\Big)^{\frac1p}.$$

Here, $(r_n)_{n=1}^N$ is a sequence of independent Rademacher random variables. For example, every Banach space has type 1, Hilbert spaces have type 2, and $L^p(S,\mu)$, with $1 \leqslant p < \infty$ has type $\min\{p,2\}$. If X has type p, then $L^r(S,\mu;X)$ has type $\min\{r,p\}$. We refer to [**?**] for more details.

When E also has type 2, then the conditions (G1) and (G2) are implied by the 'classical' notions of Lipschitz continuity and convergence assumptions, respectively, with respect to the norm of $\gamma(H, E_{-\kappa_G})$; see [**?**, Lemma 5.2] (cf. the statement of Proposition 3.8).

For UMD spaces E, under the above assumptions the existence of a unique maximal solution $(X_n(t))_{t\in[0,\sigma_n)}$ of (SCP) with coefficients A_n, F_n, G_n was proved in [**?**, Theorem 8.1] for initial data $\xi_n \in L^p(\Omega,\mathscr{F}_0,\mathbb{P};E_\theta)$ with $2 < p < \infty$. Moreover, it was shown that σ_n is an explosion time for X_n. In this context we shall write

$$X_n = \mathrm{sol}(A_n, F_n, G_n, \xi_n).$$

In the special case when the coefficients F_n and G_n are of linear growth and satisfy global Lipschitz assumptions (so that $\sigma_n \equiv T$), the convergence results proved in

[**?**, Theorems 4.3, 4.7] for the case $\theta = \kappa_F = \kappa_G = \delta = 0$ can be extended *mutatis mutandis* to yield the following result.

Proposition 3.2. *Let E be a UMD space, assume (A1), (A2), (A3), suppose the mappings $F_n : [0,T]\times\Omega\times E_\theta \to E_{-\kappa_F}$ and $G_n : [0,T]\times\Omega\times E_\theta \to \gamma(H, E_{-\kappa_G})$ satisfy the global Lipschitz counterparts of (F1), (G1) with linear growth assumptions, and assume that they satisfy (F2), (G2). Let $2 < p < \infty$, $0 \leqslant \theta < \frac{1}{2}$, $0 \leqslant \kappa_F, \kappa_G < \frac{1}{2}$ satisfy*

$$\theta + \kappa_F < \tfrac{3}{2} - \tfrac{1}{\tau}, \quad \theta + \kappa_G < 1 - \tfrac{1}{p} - \tfrac{1}{\tau}, \tag{3.1}$$

where $\tau \in (1,2]$ denotes the type of E.

Then, if $\xi_n \to \xi_\infty$ in $L^p(\Omega, \mathscr{F}_0, \mathbb{P}; E_\theta)$, then the global solutions $(X_n)_{t\in[0,T]}$ of (SCP) *satisfy $X_n \to X_\infty$ in $L^q(\Omega; C([0,T]; E_\theta))$ for all $1 \leqslant q < p$. Moreover, if $\lambda, \delta \geqslant 0$ satisfy*

$$\lambda + \delta < \tfrac{1}{2} - \tfrac{1}{p} - \kappa_G, \tag{3.2}$$

then $X_n - S_n(\cdot)\xi_n \to X_\infty - S_\infty(\cdot)\xi_\infty$ in $L^q(\Omega; C^\lambda([0,T]; E_\delta))$ for all $1 \leqslant q < p$ in case.

Proof. (Sketch) The convergence in $L^q(\Omega; C([0,T]; E_\theta))$ will follow if we prove, for some $\alpha \in (0, \frac{1}{2})$, convergence in the space $V^q_\alpha([0,T]\times\Omega; E_\theta)$ introduced in [**?**] (where it is used to prove existence and uniqueness of solutions by means of a fixed point argument). For this we can use the same strategy as in [**?**, Theorem 4.3]. First, [**?**, Lemma 4.4] is extended to our more general situation involving fractional domain spaces using Lemma A.1. Subsequently one proves that the terms considered in [**?**, Lemma 4.5] converge in $L^q(\Omega; C^\mu([0,T]; E_\theta)$ from some $\mu > \frac{1}{\tau} - \frac{1}{2}$. For example, for the terms involving the stochastic convolutions with G and G_n we can use the estimate of [**?**, Proposition 4.2] if we assume $\mu + \kappa_G + \theta < \alpha - \frac{1}{p}$. Choosing α close to $\frac{1}{2}$, we obtain the condition $\mu < \frac{1}{2} - \frac{1}{p} - \kappa_G - \theta$. Thus, to be able to choose an appropriate μ, we have to have $\frac{1}{\tau} - \frac{1}{2} < \frac{1}{2} - \frac{1}{p} - \kappa_G - \theta$, or equivalently, $\theta + \kappa_G < 1 - \frac{1}{p} - \frac{1}{\tau}$. Likewise (cf. the proof of [**?**, Theorem 6.3]), the convolutions with F and F_n can be handled if we can choose λ to satisfy $\frac{1}{\tau} - \frac{1}{2} < \mu < 1 - \kappa_F - \theta$; this is possible if $\theta + \kappa_F < \frac{3}{2} - \frac{1}{\tau}$.

The second assertion is proved similarly, following the proof of [**?**, Theorem 4.7]. □

Remark 3.3. In situations where one has $\theta = \kappa_F = \kappa_G = \delta = 0$ with F and G not necessarily globally Lipschitz continuous, Assumption (A3) is not needed in Proposition 3.2 and also not in the following results.

Using the results of the previous section, we can now extend these results to measurable initial data.

Corollary 3.4. *Let E be a UMD space and assume (A1), (A2), (A3). Moreover, let F_n and G_n as in Proposition 3.2 and assume that the coefficients $0 \leqslant \theta, \kappa_G, \kappa_G < \frac{1}{2}$ satisfy*

$$0 \leqslant \kappa_F < \tfrac{3}{2} - \tfrac{1}{\tau}, \quad \theta + \kappa_G < 1 - \tfrac{1}{\tau}$$

where τ is the type of E.

Then if $\xi_n \to \xi_\infty$ in $L^0(\Omega, \mathscr{F}_0, \mathbb{P}; E_\theta)$, the global solutions $(X_n)_{t\in[0,T]}$ of (SCP) *satisfy $X_n \to X_\infty$ in $L^0(\Omega; C([0,T]; E_\theta))$. Moreover, for $\lambda, \delta \geqslant 0$ with $\lambda + \delta < \frac{1}{2} - \kappa_G$, we have $X_n - S_n(\cdot)\xi_n \to X_\infty - S_\infty(\cdot)\xi_\infty$ in $L^0(\Omega; C^\lambda([0,T]; E_\delta))$.*

Proof. The assumptions of Theorem 2.1 are satisfied with $\rho_n^{(r)} := \inf\{t \in (0,T) : \|X_n(t)\|_\theta > r\}$ and $X_n^{(r)} = \mathrm{sol}(A_n, F_n, G_n, \xi_n^{(r)})$, where $\xi_n^{(r)} = \xi_n \mathbb{1}_{\{\|\xi_n\|_\theta \leqslant r\}}$. Noting that $\xi_n^{(r)} \to \xi_\infty^{(r)}$ in L^p for all $p > 2$, we see that for large enough $p > 2$ the

assumptions of Proposition 3.2 are satisfied for fixed r and therefore condition (b) preceding Theorem 2.1 is verified. Condition (a) is a consequence of the construction of solutions with measurable initial values, see [**?**, Section 7]. Now Corollary 2.5 yields the claim.

Similarly,the second claim follows from estimate

$$\mathbb{E}\|X_n \mathbb{1}_{\{\rho_\infty^{(N)}=T\}}\|^p_{C^{\lambda+\varepsilon}([0,T];E_\theta)} \leqslant \mathbb{E}\|X_n \mathbb{1}_{\{\rho_n^{(N)}=T\}}\|^p_{C^{\lambda+\varepsilon}([0,T];E_\theta)} \leqslant C < \infty$$

for all $n \in \mathbb{N}$, [**?**, Lemma 4.2] and a diagonal argument. $\square$

Combining this result with Theorem 2.1, we obtain the following extension of Proposition 3.2 to the locally Lipschitz case.

Theorem 3.5. *Let E be a UMD space, assume (A1), (A2), (A3), (F1), (F2), (G1), (G2), and let* (3.1) *hold. Suppose that $\xi_n \to \xi_\infty$ in $L^0(\Omega, \mathscr{F}_0, \mathbb{P}; E_\theta)$. Let $(X_n(t))_{t\in[0,\sigma_n)} = \mathrm{sol}(A_n, F_n, G_n, \xi_n)$ and define*

$$\rho_n^{(r)} := \inf\big\{t \in (0, \sigma_n) : \|X_n(t)\|_\theta > r\big\}.$$

Then,

(1) *For all $r > 0$ and $\varepsilon > 0$ we have, almost surely,*
$$\liminf_{n\to\infty} \rho_n^{(r)} \leqslant \rho_\infty^{(r)} \leqslant \limsup_{n\to\infty} \rho_n^{(r+\varepsilon)}\,;$$
(2) *For all $r > 0$ and $\varepsilon > 0$ we have*
$$X_n \mathbb{1}_{[0,\rho_\infty^{(r)}\wedge\rho_n^{(r+\varepsilon)})} \to X_\infty \mathbb{1}_{[0,\rho_\infty^{(r)})} \quad \text{in } L^0(\Omega; B_{\mathrm{b}}([0,T]; E_\theta))\,;$$
(3) *We have*
$$X_n \mathbb{1}_{[0,\sigma_\infty\wedge\sigma_n)} \to X_\infty \mathbb{1}_{[0,\sigma_\infty)} \quad \text{in } L^0(\Omega \times [0,T]; E_\theta).$$

Proof. For $r > 0$, define

$$F_n^{(r)}(t,\omega,x) := \begin{cases} F_n(t,\omega,x) & \text{if } \|x\|_\theta \leqslant r \\ F_n\big(t,\omega,\frac{rx}{\|x\|_\theta}\big) & \text{otherwise,}\end{cases}$$

and define $G_n^{(r)}$ analogously. For each $r > 0$, the maps $F_n^{(r)}$ and $G_n^{(r)}$ are uniformly (γ-)Lipschitz continuous and of linear growth. In particular, the processes $X_n^{(r)} := \mathrm{sol}(A_n, F_n^{(r)}, G_n^{(r)}, \xi_n)$ exist globally. Then the processes X_n together with the processes $X_n^{(r)}$ satisfy the hypotheses of Theorem 2.1. Indeed, (a) follows from the maximality of X_n, cf. [**?**, Lemma 8.2], and (b) follows from the convergence $X_n \to X_\infty$ in $L^0(\Omega; C([0,T]; E_\theta))$ of Proposition 3.2. $\square$

In what follows we shall always only consider the case of convergence of initial data $\xi_n \to \xi$ in $L^p(\Omega; \mathscr{F}_0, \mathbb{P}; E_\theta)$, since this case already contains the heart of the matter and suffices for the applications below. The results we present have extension to measurable initial data converging in $L^0(\Omega; \mathscr{F}_0, \mathbb{P}; E_\theta)$ which can be deduced from the L^p results as in the proof of Corollary 3.4.

Corollary 3.6. *Let the assumptions of the previous theorem be satisfied and suppose that $\xi_n \to \xi$ in $L^p(\Omega; \mathscr{F}_0, \mathbb{P}; E_\theta)$. Suppose furthermore that $\sigma_n = T$ almost surely for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \mathbb{E}\|X_n\|^p_{C([0,T];E_\theta)} < \infty$. Then:*

(1) *$\sigma_\infty = T$ almost surely;*
(2) *For all $1 \leqslant q < p$,*
$$X_n \to X_\infty \quad \text{in } L^q(\Omega; C([0,T]; E_\theta));$$
(3) *For $0 \leqslant \delta < \frac{1}{2} - \frac{1}{p} - \kappa_G$ we have*
$$X_n - S_n(\cdot)\xi_n \to X_\infty - S_\infty(\cdot)\xi_\infty \quad \text{in } L^0((0,T) \times \Omega; E_\delta);$$

(4) *If, in addition,* (3.2) *holds and* $\sup_n \mathbb{E}\|X_n - S_n(\cdot)\xi_n\|^p_{C^\lambda([0,T];E_\delta)} < \infty$*, then*
$$X_n - S_n(\cdot)\xi_n \to X_\infty - S_\infty(\cdot)\xi_\infty \quad in \ L^q(\Omega; C^\mu([0,T], E_\delta))$$
for all $1 \leqslant q < p$ *and* $0 \leqslant \mu < \lambda$.

Proof. (1) and (2) follow from Corollary 2.6.

(3) Before we start the proof we note that the result follows trivially (with convergence in a stronger sense) from (2) when $\delta \leqslant \theta$. The point of (3) is that we might have $\delta > \theta$, and this is what we shall assume in the rest of the proof.

The processes $Y_n := X_n - S_n(\cdot)\xi_n$ belong to $L^0(\Omega; C_{\mathrm{b}}([0,T); E_\delta))$ in view of $\sigma_n = T$ and [**?**, Theorem 8.1].

We first additionally assume that the initial values ξ_n are uniformly bounded in $L^\infty(\Omega, \mathscr{F}_0, \mathbb{P}; E_\theta)$ and put
$$C := \sup_{t\in[0,T], n\in\mathbb{N}} \|S_n(t)\xi_n\|_{L^\infty(\Omega;E_\theta)} .$$
Put $Z_n^{(r)} := \mathrm{sol}(A_n, F_n^{(r+C)}, G_n^{(r+C)}, \xi_n)$, where $F_n^{(r)}$ and $G_n^{(r)}$ are as in the proof of Theorem 3.5, and $Y_n^{(r)} := Z_n^{(r)} - S_n(\cdot)\xi_n$. With $\varrho_n^{(r)} := \inf\{t > 0 : \|Y_n(t)\|_\delta > r\}$ we have $Y_n \mathbb{1}_{[0,\varrho_n^{(r)}]} = Y_n^{(r)} \mathbb{1}_{[0,\varrho_n^{(r)}]}$. Indeed, if $t \leqslant \varrho_n^{(r)}$, then $\|Y_n(t)\|_\delta \leqslant r$ and
$$\|Z_n^{(r)}(t)\|_\theta \leqslant \|Y_n^{(r)}(t)\|_\theta + \|S_n(t)\xi_n\|_\theta \leqslant \|Y_n^{(r)}(t)\|_\delta + C \leqslant r + C$$
almost surely. By the maximality of X_n, $X_n \mathbb{1}_{[0,\varrho_n^{(r)}]} = Z_n^{(r)} \mathbb{1}_{[0,\varrho_n^{(r)}]}$. Subtracting $S_n(\cdot)\xi_n$, it follows that $Y_n \mathbb{1}_{[0,\varrho_n^{(r)}]} = Y_n^{(r)} \mathbb{1}_{[0,\varrho_n^{(r)}]}$ as claimed.

This proves that Hypothesis (a) preceding the statement of Theorem 2.1 is satisfied. Hypothesis (b) follows from Proposition 3.2. Thus the assertion follows from Theorem 2.1(3).

It remains to remove the additional boundedness assumption. To that end, fix $K \in \mathbb{N}$. From any given subsequence of ξ_n we can extract a further subsequence, relabeled with indices n, such that $\xi_n \to \xi_\infty$ almost surely and $\|\xi_n - \xi_\infty\|_{L^p(\Omega;E_\theta)} \leqslant 2^{-n}$. By the Chebyshev inequality, $\mathbb{P}(\|\xi_n - \xi_\infty\|_\theta > 1) \leqslant 2^{-np}$.

Now define $\Omega_K^N := \{\|\xi_n\|_\theta \leqslant K + 1 \, \forall \, n \geqslant N\}$. Then
$$\mathbb{P}(\complement\Omega_K^N) \leqslant \mathbb{P}(\|\xi_\infty\|_\theta > K) + 2^{-Np} .$$
Setting $\xi_n^{(K)} := \xi_n \mathbb{1}_{\{\|\xi_n\|_\theta \leqslant K+1\}}$, it follows that $\xi_n^{(K)} \to \xi_\infty^{(K)}$ in $L^p(\Omega; E_\theta)$ and $\xi_n^{(K)}$ is bounded (with respect to n) in $L^\infty(\Omega; E_\theta)$. By the above, the claim holds true for the processes $Y_n^{(K)}$, which are defined as the processes Y_n, but starting the uncompensated solution at the modified initial data $\xi_n^{(K)}$.

By [**?**, Lemma 8.2], almost surely on Ω_K^N, we have $Y_n^{(K)} = Y_n$. Thus along our subsequence, (2) hold with Ω replaced with Ω_K^N for all $K, N \in \mathbb{N}$. Writing Ω as a countable union of such sets, it follows that (2) holds as stated.

(4) is immediate from (2) and [**?**, Lemma 4.2]. □

Example 3.7. The condition $\sup_{n\in\mathbb{N}} \mathbb{E}\|X_n\|^p_{C([0,T];E_\theta)} < \infty$ is satisfied if, in addition to the assumptions in Theorem 3.5, F_n and G_n are uniformly of linear growth. For $\lambda, \delta \geqslant 0$ with $\lambda + \delta < \frac{1}{2} - \frac{1}{p} - \kappa_G$, we also have $\sup_n \mathbb{E}\|X_n - S_n(\cdot)\xi_n\|^p_{C^\lambda([0,T];E_\delta)} < \infty$; see [**?**, Theorem 8.1]. Hence, in this situation, Corollary 2.6(4) applies.

3.4. **Stochastic evolution equations on general Banach spaces.** Reaction diffusion type equations with nonlinearities of polynomial growth are usually considered in spaces of continuous functions. This is essential in order to verify the assumptions posed on the nonlinearities. As far as we know, there is no satisfying theory of stochastic integration available in spaces of continuous functions. We get around this by assuming that the Banach space B in which we seek the solutions is

sandwiched between E_θ and E. We then assume that E is a UMD Banach space as in the previous section and carry out all stochastic integrations in the interpolation scale of E. In order to be able to handle initial values with values in B without losing regularity due to the various embeddings, however, we need to carry out all fixed point arguments in the space $L^p(\Omega; C([0,T]; B))$.

In applications, typical choices are $B = C(\overline{\mathscr{O}})$ and $E = L^p(\mathscr{O})$ for some large $p \geqslant 2$, with $\mathscr{O}$ a domain in $\mathbb{R}^d$. This motivates us to work in UMD spaces E with type 2 from the onset (these include the spaces $L^p(\mathscr{O})$ for $2 \leqslant p < \infty$). Accordingly we shall assume:

(E) E is a UMD Banach space with type 2.

In addition to (A1) – (A3) we shall assume:

(A4) The semigroups S_n restrict to strongly continuous semigroups S_n^B on B which are uniformly exponentially bounded in the sense that, for certain constants $\tilde{M} \geqslant 1$ and $\tilde{w} \in \mathbb{R}$ we have $\|S_n(t)\|_{\mathscr{L}(B)} \leqslant \tilde{M}e^{\tilde{w}t}$ for all $t \geqslant 0$ and $n \in \overline{\mathbb{N}}$.

(A5) We have continuous, dense embeddings $E_\theta \hookrightarrow B \hookrightarrow E$.

Strong resolvent convergence of the parts $A_n|_B$ of A_n in B follows from (A1) – (A4); see Lemma A.2.

In the applications we have in mind, the operators A_n are second order elliptic differential operators on $E := L^p(\mathscr{O})$ subject to suitable boundary conditions (b.c.), where $\mathscr{O} \subseteq \mathbb{R}^d$ is some domain, and $E_\theta = H^{2\theta,p}_{\text{b.c.}}(\mathscr{O})$ is the corresponding Sobolev space. If $p \geqslant 2$ and $\theta \geqslant 0$ are chosen appropriately in relation to the dimension d, then E_θ is continuously and densely embedded into $B := C_{\text{b.c.}}(\overline{\mathscr{O}})$.

In the present framework we can repeat the procedure of the previous subsection to obtain convergence to maximal solutions of (SCP) with nonlinearities F and G which are locally Lipschitz continuous from a corresponding convergence result for globally Lipschitz continuous coefficients. In particular, the results of Theorem 3.5 and Corollary 2.6 (1) and (2) generalise *mutatis mutandis* to the situation considered here. Instead of spelling out the details we content ourselves with the statement of the convergence result for the globally Lipschitz case.

Proposition 3.8. *Let B be a Banach space, assume (E) and (A1)–(A5), and assume that* (3.1) *holds with $\tau = 2$, i.e., $2 < p < \infty$, $0 \leqslant \theta < \frac12$, $0 \leqslant \kappa_G < \frac12$ satisfy*

$$\theta + \kappa_G < \tfrac12 - \tfrac1p.$$

Moreover, let $F_n : [0,T] \times \Omega \times B \to E_{-\kappa_F}$ and $G_n : [0,T] \times \Omega \times B \to \gamma(H, E_{-\kappa_G})$ be strongly measurable, adapted, and globally Lipschitz continuous in the third variable, uniformly with respect to the first and second variables. If

$$\lim_{n\to\infty} F_n(t,\omega,x) = F_\infty(t,\omega,x) \quad \textit{and} \quad \lim_{n\to\infty} G_n(t,\omega,x) = G(t,\omega,x)$$

for all $(t,\omega,x) \in [0,T] \times \Omega \times B$ and $\xi_n \to \xi_\infty$ in $L^p(\Omega, \mathscr{F}_0, \mathbb{P}; B)$, then:

(1) *For each $n \in \overline{\mathbb{N}}$, the problem* (SCP) *with coefficients (A_n, F_n, G_n) and initial datum ξ_n has a unique mild solution X_n in $L^p(\Omega; C([0,T]; B))$;*

(2) *For all $1 \leqslant q < p$,*

$$X_n \to X_\infty \quad \textit{in } L^q(\Omega; C([0,T]; B)).$$

(3) *If $\lambda, \delta \geqslant 0$ satisfy $\lambda + \delta < \frac12 - \frac1p - \kappa_G$ then*

$$X_n - S_n(\cdot)\xi_n \to X_\infty - S(\cdot)\xi_\infty \quad \textit{in} \quad L^q(\Omega; C^\lambda([0,T]; E_\delta))$$

for all $1 \leqslant q < p$.

Note that the condition $\theta + \kappa_F < 1$, which also results from (3.1) if we take $\tau = 2$, is automatically satisfied in view of the standing assumptions $0 \leqslant \theta, \kappa_F < \frac12$.

Sketch of proof. Towards (1), let $V_T := L^p_{\mathbb{F}}(\Omega; C([0,T]; B))$ denote the space of continuous, adapted B-valued processes ϕ such that $\|\phi\|^p_{V_T} := \mathbb{E}\|\phi\|^p_{C([0,T];B)} < \infty$. By (A4), $S_n(\cdot)\xi_n \in V_T$.

Consider the fixed point operators $\Lambda_{n,\xi_n,T}$ from V_T into itself defined by

$$\big[\Lambda_{n,\xi_n,T}\phi\big](t) := S_n(t)\xi_n + S_n * F_n(\cdot,\phi)(t) + S_n \diamond G_n(\cdot,\phi)(t),$$

where

$$S * f(t) := \int_0^t S(t-s)f(s)\,ds$$

and

$$S \diamond g(t) := \int_0^t S(t-s)g(s)\,dW_H(s)$$

denote the convolution and stochastic convolution, respectively. Using [**?**, Lemma 3.4], we see that $S_n * F_n(\cdot,\phi)$ is in $L^p_{\mathbb{F}}(\Omega; C([0,T]; E_\theta))$, and hence in V_T, for all $\phi \in V_T$. Moreover, by the assumptions on G_n, we see that $s \mapsto S_n(t-s)G_n(s,\phi(s))$ is in $L^p(\Omega; L^2(0,t;\gamma(H,E_\theta)))$. Since E_θ, being isomorphic to E, is UMD with type 2, this function is stochastically integrable in E_θ. In fact, using [**?**, Proposition 4.2] one finds that the stochastic convolution $S_n \diamond G_n(\cdot,\phi)$ defines an element of $L^p_{\mathbb{F}}(\Omega; C([0,T]; E_\theta))$, and hence of V_T.

Standard arguments show that for each n, $\Lambda_{n,\xi_n,T}$ is Lipschitz continuous on V_T and the Lipschitz constants of $\Lambda_{n,\xi_n,T}$ converge to 0 as $T \downarrow 0$. Hence, for small enough T, solutions of (SCP) can be obtained from Banach's fixed point theorem and global solutions of (SCP) can be 'patched together' inductively from solutions on smaller time intervals.

(2) As in the proof of [**?**, Theorem 4.3] it suffices to prove that $\Lambda_{n,\xi_n,T}\phi \to \Lambda_{\infty,\xi_\infty,T}\phi$ in V_T for all $\phi \in V_T$ with T small. Convergence of $S_n(\cdot)\xi_n \to S_\infty(\cdot)\xi_\infty$ follows from Lemma A.2. As for the stochastic and deterministic convolutions, as in [**?**, Lemma 4.5] one sees that they actually converge in $L^p(\Omega; C([0,T]; E_\theta))$, and hence in $L^p(\Omega; C([0,T]; B))$ by (A5).

(3) follows similarly as in the proof of [**?**, Theorem 4.7]. □

4. Global existence for reaction diffusion type equations

In this section, we shall make additional assumptions on the coefficients similar to those considered by Brzeźniak and Gątarek [**?**] and Cerrai [**?**].

Throughout this section we shall assume that B is a Banach space and that E is a UMD space with type 2. *Unless explicitly stated otherwise all norms* $\|\cdot\|$ *are taken in* B.

Let us first recall that in a Banach space B, the *subdifferential of the norm at* x is given by

$$\partial\|x\| := \big\{x^* \in B^* \,:\, \|x^*\| = 1 \quad\text{and}\quad \langle x, x^*\rangle = 1\big\}.$$

We recall, see [**?**, Proposition D.4], that if $u : I \to B$ is a differentiable function, then $\|u(\cdot)\|$ is differentiable from the right and from the left with

$$\frac{d^+}{dt}\|u(t)\| = \max\big\{\big\langle u'(t), x^*\big\rangle \,:\, x^* \in \partial\|u(t)\|\big\},$$
$$\frac{d^-}{dt}\|u(t)\| = \min\big\{\big\langle u'(t), x^*\big\rangle \,:\, x^* \in \partial\|u(t)\|\big\}.$$

Since $\|u(\cdot)\|$ is everywhere differentiable from the left and from the right, it follows from [**?**, Theorem 17.9] that $\|u(\cdot)\|$ is differentiable, except for at most countably many points, and at each point t of differentiability we have

$$\frac{d}{dt}\|u(t)\| = \big\langle u'(t), x^*\big\rangle \quad\text{for all}\ \ x^* \in \partial\|u(t)\|.$$

It now follows from [**?**, Exercise 18.41] that if $t \mapsto \langle u'(t), x_t^* \rangle$ is integrable on I for suitable (equivalently, all) choices $x_t^* \in \partial\|u(t)\|$, in particular if $t \mapsto \|u(t)\|$ is integrable on I, then $t \mapsto \|u(t)\|$ is absolutely continuous and

$$\|u(t)\| = \|u(s)\| + \int_s^t \langle u'(r), x_r^* \rangle \, dr$$

for $s, t \in I$ with $s < t$ and $x_t^* \in \partial\|u(t)\|$.

Throughout this section the following standing assumptions will be in place. We assume that E is a UMD space with type 2 and suppose that A satisfies (A1), i.e., A is the generator of a strongly continuous and analytic semigroup S on E. Furthermore, we assume that (A4) and (A5) are satisfied and that S^B is a strongly continuous contraction semigroup on B. In particular, $A|_B$ is dissipative. Concerning the maps F and G we make the following assumptions.

(F$'$) The map $F : [0,T] \times \Omega \times B \to B$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_F^{(r)}$ such that

$$\|F(t,\omega,x) - F(t,\omega,y)\| \leqslant L_F^{(r)} \|x - y\|$$

for all $\|x\|, \|y\| \leqslant r$ and $(t,\omega) \in [0,T] \times \Omega$ and there exists a constant $C_{F,0} \geqslant 0$ such that

$$\|F(t,\omega,0)\| \leqslant C_{F,0}$$

for all $t \in [0,T]$ and $\omega \in \Omega$. Moreover, for all $x \in B$ the map $(t,\omega) \mapsto F(t,\omega,x)$ is strongly measurable and adapted.

For suitable constants $a', b' \geqslant 0$ and $N \geqslant 1$ we have

$$\langle Ax + F(t, x+y), x^* \rangle \leqslant a'(1 + \|y\|)^N + b'\|x\|$$

for all $x \in \mathsf{D}(A|_B)$, $y \in B$, and $x^* \in \partial\|x\|$.

(G$'$) The map $G : [0,T] \times \Omega \times B \to \gamma(H, E_{-\kappa_G})$ is locally Lipschitz continuous in the sense that for all $r > 0$ there exists a constant $L_G^{(r)}$ such that

$$\|G(t,\omega,x) - G(t,\omega,y)\|_{\gamma(H,E_{-\kappa_G})} \leqslant L_G^{(r)} \|x - y\|$$

for all $\|x\|, \|y\| \leqslant r$ and $(t,\omega) \in [0,T] \times \Omega$ and there exists a constant $C_{G,0} \geqslant 0$ such that

$$\|G(t,\omega,0)\|_{\gamma(H,E_{-\kappa_G})} \leqslant C_{G,0}$$

for all $t \in [0,T]$ and $\omega \in \Omega$. Moreover, for all $x \in B$ and $h \in H$ the map $(t,\omega) \mapsto G(t,\omega,x)h$ is strongly measurable and adapted.

Finally, for suitable constants $c \geqslant 0$ and $\varepsilon > 0$ we have

$$\|G(t,\omega,x)\|_{\gamma(H,E_{-\kappa_G})} \leqslant c'(1 + \|x\|)^{\frac{1}{N} + \varepsilon}$$

for all $(t,\omega,x) \in [0,T] \times \Omega \times B$.

Remark 4.1. In the results to follow, the constant ε in (G$'$) has to be sufficiently small.

Example 4.2. Let $B = C(\overline{\mathscr{O}})$ for some bounded domain $\mathscr{O} \subset \mathbb{R}^d$. Let $F : [0,T] \times \Omega \times B \to B$ be given by

$$(F(t,\omega,x))(s) = f(t,\omega,s,x(s)),$$

where

$$f(t,\omega,s,\eta) = -a(t,\omega,s)\eta^{2k+1} + \sum_{j=0}^{2k} a_j(t,\omega,s)\eta^j, \quad \eta \in \mathbb{R}. \tag{4.1}$$

We assume that there are constants $0 < c \leqslant C < \infty$ such that (cf. [**?**])

$$c \leqslant a(t,\omega,s) \leqslant C, \quad |a_j(t,\omega,s)| \leqslant C \quad (j = 0, \dots, 2k)$$

for all $(t,\omega,s) \in [0,T]\times\Omega\times\overline{\mathscr{O}}$. It is easy to see that, in this situation, for a suitable constant $a' \geqslant 0$ we have

$$-a'(1+|\eta|^{2k+1}\mathbb{1}_{\{\eta\geqslant 0\}}) \leqslant f(t,\omega,s,\eta) \leqslant a'(1+|\eta|^{2k+1}\mathbb{1}_{\{u\leqslant 0\}})$$

for all $t\in[0,T], \omega\in\Omega, s\in\mathscr{O}, \eta\in\mathbb{R}$. This, in turn, yields that

$$f(t,\omega,s,\eta+\zeta)\cdot\operatorname{sgn}\eta \leqslant a'(1+|\zeta|^{2k+1})$$

for all $(t,\omega,s)\in[0,T]\times\Omega\times\overline{\mathscr{O}}$ and $\eta,\zeta\in\mathbb{R}$. By the results of [**?**, Section 4.3] this implies

$$\langle F(t,\omega,x+y),x^*\rangle \leqslant a'(1+\|y\|^{2k+1})$$

for all $t\in[0,T]$, $\omega\in\Omega$, $x,y\in B$, and $x^*\in\partial\|x\|$. Since $A|_B$ is dissipative, it follows that (F′) holds.

The first main result of this section reads as follows.

Theorem 4.3. *Let B be a Banach space, assume (E), (A1), (A4), (A5), (F′), and (G′) with $\varepsilon>0$ sufficiently small, and assume that $2<p<\infty$, $0\leqslant\theta<\frac12$, $0\leqslant\kappa_F,\kappa_G<\frac12$, and*

$$\theta+\kappa_G<\tfrac12-\tfrac{1}{Np}.$$

For all $\xi\in L^p(\Omega,\mathscr{F}_0,\mathbb{P};B)$, the maximal solution $(X(t))_{t\in[0,\sigma)}$ of (SCP) *is global, i.e., we have $\sigma=T$ almost surely. Moreover,*

$$\mathbb{E}\|X\|^p_{C([0,T];B)} \leqslant C(1+\mathbb{E}\|\xi\|^p),$$

where the constant C depends on the coefficients only through the sectoriality constants of A and the constants a',b',c' and the exponent N.

This result improves corresponding results in [**?**, **?**] under similar assumptions on F and G. In [**?**], global existence of a martingale solution was obtained for uniformly bounded G; in [**?**], rather restrictive simultaneous diagonalisability assumptions on A and the noise were imposed.

In the proof of Theorem 4.3 we will use the following lemma, which is a straightforward generalisation of [**?**, Lemma 4.2]. For the reader's convenience we include the short proof.

Lemma 4.4. *Let A be the generator of a strongly continuous contraction semigroup S on B, $x\in B$ and $F:[0,T]\times B\to B$ satisfy condition (F′). If for some $\tau>0$ two continuous functions $u,v:[0,\tau)\to B$ satisfy*

$$u(t)=S(t)x+\int_0^t S(t-s)F(s,u(s)+v(s))\,ds \quad \forall\, t\in[0,\tau),$$

then

$$\|u(t)\|\leqslant e^{b't}\Big(\|x\|+\int_0^t a'(1+\|v(s)\|)^N\,ds\Big).$$

Proof. For $n\in\mathbb{N}$, put $u_n(t):=nR(n,A)u(t)$, $x_n:=nR(n,A)x$ and $F_n(t,y):=nR(n,A)F(t,y)$. Then

$$u_n(t)=S(t)x_n+\int_0^t S(t-s)\big[F(s,u_n(s)+v(s))+r_n(s)\big]\,ds$$

where $r_n(s)=F_n(s,u(s)+v(s))-F(s,u_n(s)+v(s))$. It follows that u_n is differentiable with

$$u_n'(t)=Au_n(t)+F(t,u_n(t)+v(t))+r_n(t).$$

By the observations at the beginning of this section, for almost all $t\in(0,T)$ we have, for all $x^*\in\partial\|u_n(t)\|$,

$$\begin{aligned}\frac{d}{dt}\|u_n(t)\| &= \big\langle A(t)u_n(t)+F(s,u_n(t)+v(t))+r_n(t),x^*\big\rangle\\ &\leqslant a'(1+\|v(t)\|)^N+b'\|u_n(t)\|+\|r_n(t)\|.\end{aligned}$$

Thus, by Gronwall's lemma,

$$\|u_n(t)\| \leqslant e^{b't}\Big(\|x_n\| + \int_0^t a'(1+\|v(s)\|)^N + \|r_n(s)\|\,ds\Big)\,.$$

Since $\|nR(n,A)\| \leqslant 1$ and $nR(n,A) \to I$ strongly as $n \to \infty$, the assertion follows upon letting $n \to \infty$. ☐

Proof of Theorem 4.3. Let us first assume that (G′) is satisfied with $\varepsilon = 0$; in the proof we indicate the reason why a small $\varepsilon > 0$ can be allowed.

We define

$$F_n(t,\omega,x) := \begin{cases} F(t,\omega,x) & \text{if } \|x\| \leqslant n,\\ F\big(t,\omega,\frac{nx}{\|x\|}\big) & \text{otherwise.}\end{cases}$$

We also set $X_n := \mathrm{sol}(A, F_n, G, \xi)$.

Let us first note that for $x \in \mathsf{D}(A|_B)$, $y \in B$, and $x^* \in \partial\|x\|$, we have

$$\langle Ax + F_n(t,\omega,x+y), x^*\rangle \leqslant a'(1+\|y\|)^N + b\|x\| \tag{4.2}$$

for all $(t,\omega) \in [0,T]\times\Omega$. If $\|x+y\| \leqslant n$, then this follows directly from (F′). If $\|x+y\| > n$, then

$$\begin{aligned}\big\langle Ax + F_n(t,\omega,x+y), x^*\big\rangle &= \Big\langle Ax + F\big(t,\omega,\frac{nx}{\|x+y\|} + \frac{ny}{\|x+y\|}\big), x^*\Big\rangle \\ &= \Big\langle A\frac{nx}{\|x+y\|} + F\big(t,\omega,\frac{nx}{\|x+y\|} + \frac{ny}{\|x+y\|}\big), x^*\Big\rangle \\ &\quad + (1 - \frac{n}{\|x+y\|})\langle Ax, x^*\rangle \\ &\leqslant a'\Big(1 + \Big(\frac{n\|y\|}{\|x+y\|}\Big)\Big)^N + b'\Big\|\frac{n\|y\|}{\|x+y\|}\Big\| \\ &\leqslant a'(1+\|y\|^N) + b'\|x\|\end{aligned}$$

where we have used (F′) and the dissipativity of A in B in the third step and $\|x+y\| > n$ in the fourth.

Trivially,

$$\begin{aligned}&\mathbb{E}\|X_n\|^p_{C([0,T];B)} \\ &\qquad \lesssim \mathbb{E}\|S(\cdot)\xi + S * F_n(\cdot,X_n)\|^p_{C([0,T];B)} + \mathbb{E}\|S \diamond G(\cdot,X_n)\|^p_{C([0,T];B)}\,.\end{aligned} \tag{4.3}$$

By (4.2) and Lemma 4.4, applied with

$$u_n := X_n - S\diamond G(\cdot,X_n), \quad v_n = S \diamond G(\cdot, X_n),$$

we obtain

$$\begin{aligned}&\mathbb{E}\|S(\cdot)\xi + S * F_n(\cdot,X_n)\|^p_{C([0,T];B)} \\ &\quad = \mathbb{E}\sup_{t\in[0,T]}\Big\|S(\cdot)\xi + \int_0^t S(t-s)F_n(s,u_n(s)+v_n(s))\,ds\Big\|^p \\ &\quad \leqslant e^{b'pT}\mathbb{E}\sup_{t\in[0,T]}\Big(\|\xi\| + \int_0^t a'\big(1+\|v_n(s)\|\big)^N\,ds\Big)^p \\ &\quad \lesssim e^{b'pT}\mathbb{E}\big(1 + \|\xi\|^p + \|S\diamond G(\cdot,X_n)\|^N_{C([0,T];B)}\big)^p \\ &\quad \lesssim e^{b'pT}T^p\big(1 + \mathbb{E}\|\xi\|^p + \mathbb{E}\|S\diamond G(\cdot,X_n)\|^{Np}_{C([0,T];B)}\big)\,.\end{aligned}$$

Since $\theta + \kappa_G < \frac12 - \frac{1}{Np}$, we may pick $\alpha \in (0,\frac12)$ such that $\theta + \kappa_G < \alpha - \frac{1}{Np}$. Then, for some $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{E}\|S\diamond G(\cdot,X_n)\|^{Np}_{C([0,T];B)} \\ &\qquad \lesssim \mathbb{E}\|S\diamond G(\cdot,X_n)\|^{Np}_{C([0,T];E_\theta)}\end{aligned}$$

$$
\begin{aligned}
&\lesssim T^{\varepsilon Np}\mathbb{E}\int_0^T \|s\mapsto (t-s)^{-\alpha}G(\cdot,X_n)\|^{Np}_{\gamma(L^2(0,t;H),E_{-\kappa_G})}\,dt\\
&\lesssim T^{\varepsilon Np}\mathbb{E}\int_0^T \|s\mapsto (t-s)^{-\alpha}G(\cdot,X_n)\|^{Np}_{L^2(0,t;\gamma(H,E_{-\kappa_G}))}\,dt\\
&= T^{\varepsilon Np}\mathbb{E}\int_0^T \Big(\int_0^t (t-s)^{-2\alpha}\|G(s,X_n(s))\|^2_{\gamma(H,E_{-\kappa_G})}\,ds\Big)^{\frac{Np}{2}}\,dt\\
&\overset{(*)}{\leqslant} T^{\varepsilon Np}\Big(\int_0^T t^{-2\alpha}\,dt\Big)^{\frac{Np}{2}}\mathbb{E}\int_0^T \|G(t,X_n(t))\|^{Np}_{\gamma(H,E_{-\kappa_G})}\,dt\\
&\leqslant T^{(\frac12-\alpha+\varepsilon)Np}(c')^{Np}\mathbb{E}\int_0^T (1+\|X_n(t)\|)^p\,dt\\
&\lesssim T^{(\frac12-\alpha+\varepsilon)Np+1}(c')^{Np}(1+\mathbb{E}\|X_n\|^p_{C([0,T];B)})\,.
\end{aligned}
$$

In this computation we used the following facts. The first inequality follows from the continuity of the embedding $E_\theta \hookrightarrow B$, the second from [**?**, Proposition 4.2] (here the condition on α is used), the third uses the fact that if (S,μ) is a σ-finite measure space, H a Hilbert space and F a Banach space with type 2, then we have a continuous embedding $L^2(S,\mu;\gamma(H,F)) \hookrightarrow \gamma(L^2(S,\mu;H),F)$ of norm less than or equal to the type 2 constant of F, in the next inequality we used Young's inequality, and in the sixth step the assumptions on G.

Because of the strict inequality $\alpha < \frac12$, in step $(*)$ we can apply Young's inequality with slightly sharper exponents. This creates room (explicitly computable in terms of the other exponents involved) for a small $\varepsilon > 0$ in Hypothesis (G$'$).

Combining these estimates we obtain

$$
\begin{aligned}
\mathbb{E}\|S(\cdot)\xi + &S * F_n(\cdot,X_n)\|^p_{C([0,T];B)}\\
&\lesssim e^{b'pT}T^p\big(1+\mathbb{E}\|\xi\|^p + T^{(\frac12-\alpha+\varepsilon)Np+1}(1+\mathbb{E}\|X_n\|^p_{C([0,T];B)})\big)\,.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\mathbb{E}\|S\diamond G(\cdot,X_n)\|^p_{C([0,T];B)} &\leqslant \big(\mathbb{E}\|S\diamond G(\cdot,X_n)\|^{Np}_{C([0,T];B)}\big)^{\frac1N}\\
&\lesssim T^{(\frac12-\alpha+\varepsilon)Np+1}(1+\mathbb{E}\|X_n\|^p_{C([0,T];B)})\,.
\end{aligned}
$$

Substituting these estimates into (4.3) we obtain

$$
\mathbb{E}\|X_n\|^p_{C([0,T];B)} \leqslant C_0 + C_1\mathbb{E}\|\xi\|^p + C_2(T)\mathbb{E}\|X_n\|^p_{C([0,T];B)}
$$

for a certain constants C_0, C_1 and a function $C_2(T)$ which does not depend on ξ and converges to 0 as $T\downarrow 0$. Hence, if $T>0$ is small enough, we obtain

$$
\mathbb{E}\|X_n\|^p_{C([0,T];B)} \leqslant (1-C_2(T))^{-1}(C_0 + C_1\mathbb{E}\|\xi\|^p).
$$

Iterating this procedure a finite number of times, it follows that given $T>0$, there exists a constant C as in the statement such that $\sup_n \mathbb{E}\|X_n\|^p_{C([0,T];B)} \leqslant C(1+\mathbb{E}\|\xi\|^p) < \infty$. By Corollary 2.6, the lifetime of X equals T almost surely and we have $X_n \to X$ in $L^q(\Omega;C([0,T];B))$ for all $1\leqslant q<p$. □

Our next aim is to prove a version of Theorem 4.3 (Theorem 4.9 below) which, in return for an additional assumption on F, allows nonlinearities G of linear growth. For this purpose we introduce the following hypotheses.

(F$''$) There exist constants $a'',b'',m>0$ such that the function $F:[0,T]\times\Omega\times B\to B$ satisfies

$$
\langle F(t,\omega,y+x)-F(t,\omega,y),x^*\rangle \leqslant a''(1+\|y\|)^m - b''\|x\|^m
$$

for all $t\in[0,T]$, $\omega\in\Omega$, $x,y\in B$, and $x^*\in\partial\|x\|$, and

$$
\|F(t,y)\| \leqslant a''(1+\|y\|)^m
$$

for all $y \in B$.

(G$''$) The function $G : [0,T] \times \Omega \times B \to \gamma(H, E_{-\kappa_G})$ satisfies the measurability and adaptedness assumption of (G$'$) and is locally Lipschitz continuous and of linear growth. Moreover, we have

$$\|G(t,\omega,0)\|_{\gamma(H,E_{-\kappa_G})} \leqslant C_{G,0}$$

for all $(t,\omega) \in [0,T] \times \Omega$ and a suitable constant $C_{G,0} \geqslant 0$.

Example 4.5. The map F described in Example 4.2 also satisfies condition (F$''$). Indeed, for the function f as in Example 4.2, it is easy to see that for certain constants $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 > 0$ we have

$$a_1 - b_1\eta^{2k+1} \leqslant f(t,\omega,s,\eta) \leqslant a_2 - b_2\eta^{2k+1}$$

for all $(t,\omega,s,\eta) \in [0,T] \times \Omega \times \overline{\mathscr{O}} \times \mathbb{R}$. But this yields that

$$W := \big[f(t,\omega,s,\eta+\zeta) - f(\zeta)\big] \cdot \operatorname{sgn}\eta \leqslant a - b|\eta|^{2k+1} + c|\zeta|^{2k+1} \tag{4.4}$$

for certain positive constants a, b, c and all $(t,\omega,s) \in [0,T] \times \Omega \times \overline{\mathscr{O}}$ and $\eta, \zeta \in \mathbb{R}$.

To see this, we distinguish several cases.

- $\eta, \zeta \geqslant 0$. In this case,
$$\begin{aligned} W &\leqslant a_2 - b_2(\eta+\zeta)^{2k+1} - a_1 + b_1\zeta^{2k+1} \\ &\leqslant a_2 - a_1 - b_2|\eta|^{2k+1} + b_1|\zeta|^{2k+1} \end{aligned}$$
since $\eta + \zeta \geqslant \eta = |\eta|$.
- $\eta, \zeta \leqslant 0$. In this case,
$$\begin{aligned} W &\leqslant a_2 - b_2\zeta^{2k+1} - a_1 + b_1(\eta+\zeta)^{2k+1} \\ &= a_2 - a_1 + b_2|\zeta|^{2k+1} - b_2(|\eta| + |\zeta|)^{2k+1} \\ &\leqslant a_2 - a_1 + b_2|\zeta|^{2k+1} - b_2|\eta|^{2k+1}. \end{aligned}$$
- $\eta \leqslant 0 \leqslant \zeta$. In this case,
$$\begin{aligned} W &\leqslant a_2 - b_2\zeta^{2k+1} - a_1 + b_1(\eta+\zeta)^{2k+1} \\ &= a_2 - b_2|\zeta|^{2k+1} - a_1 + b_1(|\zeta| - |\eta|)^{2k+1}. \end{aligned}$$
If $|\zeta| \geqslant |\eta|$, then this can be estimated by
$$a_2 - a_1 - b_2|\eta|^{2k+1} + b_1|\zeta|^{2k+1}\,.$$
If $0 \neq |\zeta| \leqslant |\eta|$, then
$$\begin{aligned} W &\leqslant a_2 - a_1 + b_1|\zeta|^{2k+1}\big(1 - \big|\frac{\eta}{\zeta}\big|\big)^{2k+1} \\ &\leqslant a_2 - a_1 + b_1|\zeta|^{2k+1}\left(1 - \Big|\frac{\eta}{\zeta}\Big|^{2k+1} + \sum_{j=1}^{2k}\binom{2k+1}{l}\right). \end{aligned}$$
- The case where $\zeta \leqslant 0 \leqslant \eta$ can be handled similarly.

This shows that (4.4) holds for $a = a_2 - a_1$, $b = \min\{b_1, b_2\}$ and $c = \max\{b_1, b_2\}\big(1 + \sum_{j=1}^{2k}\binom{2k+1}{l}\big)$. Now, with the same strategy as in [**?**], one infers (F$''$) from (4.4).

Following the ideas of [**?**], we proceed through the use of a comparison principle. For the reader's convenience we include the proof, which is similar to that of [**?**, §9 Satz IX].

Lemma 4.6. *Let $f : (a,b) \times (c,d) \to \mathbb{R}$ be continuous and uniformly locally Lipschitz continuous in the second variable, i.e., for all compact $K \subseteq (c,d)$ there exists a constant $L = L(K)$ such that*

$$|f(t,x) - f(t,y)| \leqslant L|x-y| \quad \forall\, x, y \in K,\ t \in (a,b)\,.$$

Suppose the functions $u^+, u^- : [\alpha, \beta] \to (c, d)$ *are absolutely continuous functions and satisfy, for almost all* $t \in (\alpha, \beta)$,

$$\frac{d}{dt} u^+(t) \geqslant f(t, u^+(t)), \quad \frac{d}{dt} u^-(t) \leqslant f(t, u^-(t)).$$

If $u^+(t_0) > u^-(t_0)$ *for some* $t_0 \in [\alpha, \beta]$, *then* $u^+(t) > u^-(t)$ *for all* $t \in [t_0, \beta]$.

Proof. We may of course assume that $t_0 \in [\alpha, \beta)$.

Put $d(t) := u^+(t) - u^-(t)$ for $t \in [\alpha, \beta]$. Suppose that $A := \{t \in (t_0, \beta] : d(t) \leqslant 0\}$ is nonempty. Then, by continuity, $t_1 := \inf A > t_0$. Moreover, $d(t) > 0$ on $[t_0, t_1)$ and $d(t_1) = 0$.

Let $K = K^+ \cup K^-$ with $K^\pm := \{u^\pm(t) : t \in [\alpha, \beta]\}$ and denote by L the corresponding Lipschitz constant from the hypothesis. For almost all $s \in (t_0, t_1)$ we have

$$\begin{aligned} d'(s) &= \frac{d}{ds}(u^+(s) - u^-(s)) \geqslant f(s, u^+(s)) - f(s, u^-(s)) \\ &\geqslant -L|u^+(s) - u^-(s)| = -Ld(s) \end{aligned}$$

since $s < t_1$. It follows that $\frac{d'}{d} \geqslant -L$ almost everywhere on (t_0, t_1) and hence, by integration, $d(t) \geqslant d(t_0)e^{-L(t-t_0)}$ for all $t \in (t_0, t_1)$. By continuity, $d(t_1) \geqslant d(t_0)e^{-L(t_1-t_0)} > 0$, which contradicts $d(t_1) = 0$. Hence we must have $A = \varnothing$ and thus $u^+(t) > u^-(t)$ for all $t \in (t_0, \beta]$ as claimed. □

Corollary 4.7. *Let* f *and* u^+, u^- *be as in Lemma 4.6 but assume now that* $u^+(t_0) \leqslant u^-(t_0)$ *for some* $t_0 \in [\alpha, \beta]$. *Then* $u^+(t) \leqslant u^-(t)$ *for all* $t \in [\alpha, t_0]$.

Proof. If $u^+(t_1) > u^-(t_1)$ for some $t_1 \in [\alpha, t_0)$, then Lemma 4.6 would imply that $u(t_0) > u^-(t_0)$. □

The next lemma should be compared with Lemma 4.4.

Lemma 4.8. *Let* A *be the generator of a strongly continuous contraction semigroup* S *on* B *and let* $F : [0, T] \times B \to B$ *satisfy conditions (F′) and (F″). If* $u, v \in C([0, T]; B)$ *satisfy*

$$u(t) = \int_0^t S(t-s)F(s, u(s) + v(s))\, ds,$$

for all $t \in [0, T]$, *then*

$$\sup_{t \in [0,T]} \|u(t)\| \leqslant \big(\frac{4a''}{b''}\big)^{\frac{1}{m}} \Big(1 + \sup_{t \in [0,T]} \|v(t)\|\Big).$$

Proof. To simplify notations we write $a = a''$ and $b = b''$, where a'', b'' are as in (F″).

Step 1 – First we assume that A is bounded. Then u is continuously differentiable and

$$u'(t) = Au(t) + F(t, u(t) + v(t)), \quad t \in [0, T].$$

By the remarks at the beginning of the section, for almost all $t \in (0, T)$ we have, for all $x^* \in \partial\|u(t)\|$,

$$\begin{aligned} \frac{d}{dt}\|u(t)\| &= \langle Au(t), x^* \rangle + \langle F(t, u(t) + v(t)) - F(t, v(t)), x^* \rangle + \langle F(t, v(t)), x^* \rangle \\ &\leqslant 0 + 2a(1 + \|v(t)\|)^m - b\|u(t)\|^m \\ &\leqslant 2a\Big(1 + \sup_{s \in [0,T]} \|v(s)\|\Big)^m - b\|u(t)\|^m . \end{aligned}$$

In the second estimate we have used the dissipativity of A and our assumptions.

Setting $\varphi(t) := \|u(t)\|$ and $\gamma := (2a)^{\frac{1}{m}}(1 + \sup_{s\in[0,T]} \|v(s)\|)$, it follows that φ is absolutely continuous with

$$\varphi'(t) \leqslant -b\varphi(t)^m + \gamma^m$$

almost everywhere. We have to prove that $\varphi(t) \leqslant \big(\frac{2}{b}\big)^{\frac{1}{m}}\gamma$ for all $t \in [0,T]$. Assume to the contrary that $\varphi(t_0) > \big(\frac{2}{b}\big)^{\frac{1}{m}}\gamma$ for some $t_0 \in [0,T]$. Clearly $\varphi(0) = 0$, so $t_0 \in (0,T]$. Let $\psi : I \to \mathbb{R}$ be the unique maximal solution of

$$\begin{cases} \psi'(t) = -b\psi(t)^m + \gamma^m, \\ \psi(t_0) = \varphi(t_0). \end{cases}$$

By Corollary 4.7, $\psi(t) \leqslant \varphi(t)$ for all $t \in I \cap [0,t_0]$.

We claim that $\psi(t) > (\frac{1}{b})^{\frac{1}{m}}\gamma$ for all $t \in I \cap [0,t_0]$. If the claim was false, noting that $\psi(t_0) = \varphi(t_0) > \big(\frac{2}{b}\big)^{\frac{1}{m}}\gamma$, we would have $\psi(t_1) = \big(\frac{1}{b}\big)^{\frac{1}{m}}\gamma$ for some $t_1 \in I\cap[0,t_0]$. By uniqueness, this would imply that $\psi \equiv (\frac{1}{b})^{\frac{1}{m}}\gamma$, a contradiction to $\psi(t_0) > (\frac{1}{b})^{\frac{1}{m}}\gamma$. This proves the claim.

We have proved that $\big(\frac{1}{b}\big)^{\frac{1}{m}}\gamma < \psi \leqslant \varphi$ on $I \cap [0,t_0]$. It follows that $0 \in I$ since otherwise ψ, and hence φ, would blow up at some point in $[0,t_0)$.

Consequently, $\big(\frac{1}{b}\big)^{\frac{1}{m}}\gamma < \psi$ on $I \cap [0,t_0]$, which implies that $\psi'(t) < 0$ and hence that ψ is decreasing. It follows that

$$0 = \varphi(0) \geqslant \psi(0) \geqslant \psi(t_0) = \varphi(t_0) > \big(\frac{2}{b}\big)^{\frac{1}{m}}\gamma,$$

a contradiction.

Step 2 – In order to remove the assumption that A is bounded, we approximate A with its Yosida approximands $A_n := nAR(n,A) = n^2R(n,A) - n$. We note that if A is dissipative, then so are all A_n. We denote the (contraction) semigroup generated A_n by S_n. Let u_n be the unique fixed point in $C([0,T];B)$ of

$$w \mapsto \Big[t \mapsto \int_0^t S_n(t-s)F(s,w(s)+v(s))\,ds\Big].$$

We note that, by the local Lipschitz assumption on F, there always exists a unique maximal solution of this equation. By Theorem 4.3 with $G \equiv 0$ this solution is global. Assumption (A3) is not needed for this part of the argument; cf. Remark 3.3.

By the above,

$$\sup_{t\in[0,T]} \|u_n(t)\| \leqslant \big(\frac{4a}{b}\big)^{\frac{1}{m}}\Big(1 + \sup_{t\in[0,T]} \|v(t)\|\Big)$$

for all $n \in \mathbb{N}$. Since $u_n \to u$ in $C([0,T];B)$, this gives the desired result. $\square$

We can now extend Theorem 4.3 assuming that G is of linear growth.

Theorem 4.9. *Assume (A1), (A4), (A5), (F′), (F″), (G″) and let $p > 2$ satisfy $\theta + \kappa_G < \frac{1}{2} - \frac{1}{Np}$. Then for all $\xi \in L^p(\Omega, \mathscr{F}_0, \mathbb{P}; B)$ the maximal solution $(X(t))_{t\in[0,\sigma)}$ of* (SCP) *is global. Moreover,*

$$\mathbb{E}\|X\|^p_{C([0,T];B)} \leqslant C(1 + \mathbb{E}\|\xi\|^p),$$

where the constant C depends on the coefficients only through the sectoriality constants of A and the constants a'', b'', c'' and the exponent N.

Proof. For $n \in \mathbb{N}$ we put

$$G_n(t,\omega,x) := \begin{cases} G(t,\omega,x), & \|x\| \leqslant n \\ G\big(t,\omega,\frac{nx}{\|x\|}\big), & \text{otherwise.} \end{cases}$$

Since G is of linear growth, G_n is bounded. In particular, A, F and G_n satisfy the Hypotheses (F′) and (G′). Hence, by Theorem 4.3, $X_n := \mathrm{sol}(A, F, G_n, \xi)$ exists globally.

Proceeding as in the proof of Theorem 4.3, we have

$$\mathbb{E}\|X_n\|^p_{C([0,T];B)} \lesssim \mathbb{E}\|\xi\|^p + \mathbb{E}\|S * F(\cdot, X_n)\|^p_{C([0,T];B)} + \mathbb{E}\|S \diamond G_n(\cdot, X_n)\|^p_{C([0,T];B)}\,. \tag{4.5}$$

Using Lemma 4.8 with

$$u_n = X_n - S(\cdot)\xi - S \diamond G(\cdot, X_n) \quad \text{and} \quad v_n = S(\cdot)\xi + S \diamond G(\cdot, X_n)$$

we obtain

$$\mathbb{E}\|S * F(\cdot, X_n)\|^p_{C([0,T];B)} \lesssim \big(\frac{4a''}{b''}\big)^{\frac{1}{m}}\big(1 + \mathbb{E}\|\xi\|^p + \mathbb{E}\|S \diamond G(\cdot, X_n)\|^p_{C([0,T];B)}\big)\,.$$

Moreover, a computation similar as in the proof of Theorem 4.3 yields

$$\mathbb{E}\|S \diamond G_n(\cdot, X_n)\|^p_{C([0,T];B)} \leqslant C(T)\big(\alpha + \beta\mathbb{E}\|X_n\|^p_{C([0,T];B)}\big)$$

where $C(T) \to 0$ as $T \to 0$ and α, β only depend on the constants in the linear growth assumption on G. Substituting this back into (4.5), it follows that

$$\mathbb{E}\|X_n\|^p_{C([0,T];B)} \leqslant C_0 + C_1\mathbb{E}\|\xi\|^p + C_2(T)\mathbb{E}\|X_n\|^p_{C([0,T];B)}$$

and the proof can be finished as that of Theorem 4.3 □

In combination with our earlier results, it can be seen that the solution X in Theorem 4.9 depends continuously on the data A, F, G, and ξ in the sense discussed in Section 3. We leave the precise statement of this result to the reader.

5. Application to reaction diffusion equations

In this section, we apply our results to stochastic reaction diffusion equations with multiplicative noise which is white in time and coloured in space; in dimension 1 the noise may also be white in space. For ease of notation, we will also only consider coefficients f and g which do not depend on ω, although this case could be covered as well at the expense of additional technicalities.

On a domain $\mathscr{O} \subseteq \mathbb{R}^d$ with C^∞-boundary we consider the stochastic partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \mathscr{A}u(t,x) + f(t,x,u(t,x)) + g(t,x,u(t,x))R\dfrac{\partial w}{\partial t}(t,x), \\ \quad u(0,x) = \xi(x), \end{cases} \tag{5.1}$$

for $(t,x) \in [0,T] \times \mathscr{O}$. Here, w is a space-time white noise on $\mathscr{O}$ (i.e. an $L^2(\mathscr{O})$-cylindrical Brownian motion) and R is the identity operator on $L^2(\mathscr{O})$ (in dimension $d = 1$), respectively, for $d \geqslant 2$ a γ-radonifying operator from $L^2(\mathscr{O})$ to $L^q(\mathscr{O})$) for a suitable exponent $q \in [2, \infty)$ to be specified below.

We supply (5.1) with Neumann type boundary conditions (see (5.2) below). Here, $\mathscr{A}$ is a second order elliptic operator, formally given by

$$\mathscr{A} = \sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j\frac{\partial}{\partial x_j} + c$$

where the coefficients a_{ij}, b_j, c are real-valued, $a_{ij} = a_{ji} \in C^1(\overline{\mathscr{O}})$ satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{ij}x_ix_j \geqslant \kappa|x|^2, \quad x \in \mathscr{O},$$

and $b_j, c \in C(\overline{\mathscr{O}})$. The boundary operator $\mathscr{B}$ is given by

$$\mathscr{B} = \sum_{i,j=1}^{d} a_{ij}\nu_i \frac{\partial}{\partial x_j} \tag{5.2}$$

where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit outer normal to $\mathscr{O}$.

Finally, the nonlinearity $f : [0,T] \times \mathscr{O} \times \mathbb{R} \to \mathbb{R}$ is as in Example 4.2 and $g : [0,T] \times \mathscr{O} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and of linear growth in the third variable, uniformly with respect to the first two variables.

Remark 5.1. Using some recent deep results in elliptic PDE, the assumptions on the operator $\mathscr{A}$ can be relaxed. As this would only distract from the point we want to make, we leave such generalisations to the interested reader.

Let us rewrite equation (5.1) in our general abstract framework. We set $E = L^q(\mathscr{O})$ with a parameter $q \in [2,\infty)$ to be specified below. Then E is a UMD Banach space with type 2, so that condition (E) is satisfied. The operator A is the realisation of $\mathscr{A}$ with boundary conditions $\mathscr{B}u = 0$ on E, i.e. the domain of A is given as $\mathsf{D}(A) = \{u \in H^{2,q}(\mathscr{O}) : \mathscr{B}u = 0 \text{ on } \partial\mathscr{O}\}$, where the boundary condition has to be understood in the sense of the trace, and $Au = \mathscr{A}u$ for $u \in \mathsf{D}(A)$. Then A generates an analytic, strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on E. Hence, if we set $A_n \equiv A$, then (A1) is satisfied. Note that (A2) and (A3) are trivially satisfied. Replacing A with $A - c$ and f with $f + c$ for a suitable constant c if necessary, we may and will assume that S is uniformly exponentially stable. In particular, we may assume that $0 \in \varrho(A)$.

For further purposes, it will be more convenient to consider complex interpolation spaces instead of fractional domain spaces. Recall, cf. [**?**], that $[E, \mathsf{D}(A)]_a \hookrightarrow \mathsf{D}((-A)^b)$ and $\mathsf{D}((-A)^a) \hookrightarrow [E, \mathsf{D}(A)]_b$ for $0 < b < a < 1$. Hence we can take for E_α in (A3) and subsequently the complex interpolation spaces of index α instead of the fractional domain spaces of index α; implicitly, we have to replace α with $\alpha \pm \varepsilon$ for a small enough ε.

Define

$$H^{s,q}_{\{\mathscr{B}\}}(\mathscr{O}) := \left\{ f \in H^{s,q}(\mathscr{O}) \,:\, \mathscr{B}f = 0 \text{ on } \partial\mathscr{O} \text{ for } 1 + \tfrac{1}{q} < s \right\}.$$

Then $\mathsf{D}(A) = H^{2,q}_{\{\mathscr{B}\}}(\mathscr{O})$. Moreover, as a consequence of [**?**, Theorem 4.1], if $\theta \in (0,1)$ and $2\theta - \frac{1}{q} \neq 1$, then

$$E_\theta := [E, \mathsf{D}(A)]_\theta = H^{2\theta,q}_{\{\mathscr{B}\}}(\mathscr{O}).$$

By Sobolev embedding, if $sq > d$, then $H^{s,q}_{\{\mathscr{B}\}}(\mathscr{O}) \hookrightarrow C(\overline{\mathscr{O}}) =: B$. Consequently, by the analyticity of $(S(t))_{t\geqslant 0}$ condition (A4) is satisfied whenever $2q > d$; if $\theta \in (0,1)$ is such that $2\theta q > d$, then also condition (A5) is satisfied.

The nonlinearity F is modeled as in the previous section, where it was seen that (F$'$) and (F$''$) hold. Concerning the stochastic term, let us first consider the case $d = 1$ where we put $R = I$. Concerning G, we first pick $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$. Following [**?**, Section 10.2] we define the multiplication operator $\Gamma : [0,T] \times B \to \mathscr{L}(H)$ by

$$[\Gamma(t,u)h](s) := g(t,s,u(s))h(s), \quad s \in \mathscr{O},$$

and then define $G : [0,T] \times B \to \gamma(H, E_{-\kappa_G})$ by

$$(-A)^{-\kappa_G} G(t,u)h := \iota\jmath(-A)^{-\kappa_G}\Gamma(t,u)h\,,$$

where $\jmath : H^{2\kappa_G,p}_{\{\mathscr{B}\}}(\mathscr{O}) \to H^{2\kappa_G,2}(\mathscr{O})$ and $\iota : H^{2\kappa_G}(D) \to L^q(D) = E$ are the canonical inclusions. Note that by [**?**, Corollary 2.2], ι is γ-radonifying. Arguing as in [**?**, Section 10.2] one sees that G takes values in $\gamma(H, E_{-\kappa_G})$ is locally Lipschitz

continuous and of linear growth as a map from $[0,T]\times B\to\gamma(H,E_{-\kappa_G})$. Thus G satisfies assumption (G$''$).

Hence, from Theorem 4.9 we obtain:

Theorem 5.2 (Reaction-diffusion equation with white noise, $d=1$)**.** *Let $d=1$ and $\frac{p}{4}>2k+1$, where k is the exponent in the reaction term* (4.1)*. Under the assumptions above, for every $\xi\in L^p(\Omega,\mathscr{F}_0,\mathbb{P};B)$ the solution X of equation* (5.1) *with $R=I$ exists globally and belongs to $L^p(\Omega;C([0,T];B))$.*

Proof. The condition $\frac{p}{4}>N$, with $N=2k+1$, allows us to choose $2\leqslant q<\infty$, $\theta\in[0,\frac12)$ and $\kappa_G\in(\frac14,\frac12)$ such that, with $E=L^q(D)$ and q so large that $2\theta q>d=1$, whence $E_\theta\hookrightarrow B$, and $0\leqslant\theta+\kappa_G<\frac12-\frac{1}{pN}$. By the above discussion, the assumptions of Theorem 4.9 are then satisfied. □

Let us now discuss the situation where $d>1$. We now assume that $R\in\gamma(H,L^q(\mathscr{O}))$ for a q as specified below. We again work on $E=L^q(\mathscr{O})$, define the multiplication operator $\Gamma_E:[0,T]\times B\to\mathscr{L}(E)$ by

$$[\Gamma_E(t,u)h](s):=g(t,s,u(s))h(s)$$

and then define $G:[0,T]\times B\to\gamma(H,E)$ by $G(t,u)h:=\Gamma_E(t,u)Rh$. It is easy to see that G defined in this way satisfies assumption (G$''$) with $\kappa_G=0$. For example, if $u,v\in B$ with $\|u\|_\infty,\|u\|_\infty\leqslant r$, then

$$\begin{aligned}\|G(t,u)-G(t,v)\|_{\gamma(H,E)}&\leqslant\|\Gamma_E(t,u)-\Gamma_E(t,v)\|_{\mathscr{L}(E)}\|R\|_{\gamma(H,E)}\\&\leqslant L_g^{(r)}\|u-v\|_\infty\|R\|_{\gamma(H,E)}\end{aligned}$$

where $L_g^{(r)}$ is the Lipschitz constant of the function g on the ball $\{x\in\mathbb{R}:|x|\leqslant r\}$.

Thus in this case, we obtain

Theorem 5.3 (Reaction-diffusion equation with coloured noise, $d\geqslant 2$)**.** *Let $d\geqslant 2$ and let $q\geqslant 2$ satisfy $\frac{d}{2q}<\frac12-\frac{1}{Np}$. Assume that $R\in\gamma(H,L^q(\mathscr{O}))$. Then for every $\xi\in L^p(\Omega,\mathscr{F}_0,\mathbb{P};B)$ the solution X of equation* (5.1) *exists globally and belongs to $L^p(\Omega;C([0,T];B))$.*

Proof. We can pick $\theta\in(\frac{d}{2q},\frac12-\frac{1}{Np})$. Then $2q>2\theta q>d$, so that (A5) is satisfied. Moreover $\theta+\kappa_G=\theta<\frac12-\frac{1}{Np}$. Since all other assumptions of Theorem 4.9 are satisfied by the above discussion, the result follows. □

Let us end this article by discussing the dependence of the solution upon the coefficients A,F and G. Suppose for every $n\in\overline{\mathbb{N}}$ we are given an operator $\mathscr{A}_n$, determined through its coefficients a_n,b_n and c_n, and functions $f_n,g_n:[0,T]\times B\to B$. Let A_n,F_n and G_n be defined by replacing $\mathscr{A},f$ and g with $\mathscr{A}_n,f_n$ and g_n, respectively.

We assume that f_n and g_n satisfy the assumptions of this section uniformly for all n. We leave it to the reader to check that the resulting maps F_n and G_n satisfy growth and Lipschitzianity conditions uniformly in n and merely discuss under which conditions our convergence assumptions are satisfied.

Then we have $F_n(t,u)\to F(t,u)$ in B, if $f_n(t,\cdot,\cdot)\to f(t,\cdot,\cdot)$ for all $t\in[0,1]$, uniformly on compact subsets of $[0,1]\times\mathbb{R}$. This is a stronger assumption than in [**?**], where only pointwise convergence was required. However, for reaction diffusion equations we need convergence in $C(\overline{\mathscr{O}})$.

To infer convergence $G_n(t,u)\to G(t,u)$ for all $t\in[0,1],u\in B$, it is sufficient to have convergence $g_n(t,x,s)\to g(t,x,s)$ for all $(t,x,s)\in[0,T]\times[0,1]\times\mathbb{R}$. Indeed, if $d=1$, then under this assumption we clearly have $\Gamma_n(t,u)h\to\Gamma(t,u)h$ in $L^2(\mathscr{O})$ for all $t\in[0,T]$ and $u\in B$. Hence, by 'convergence by right multiplication', see [**?**, Proposition 2.4], convergence of $G_n(t,u)\to G(t,u)$ in $\gamma(H,E_{-\kappa_G})$ follows. In the case where $d\geqslant 2$ we obtain convergence conveniently by 'γ-dominated convergence'

[**?**, Corollary 9.4], noting that in this case, for fixed $u \in B$ and $t \in [0,t]$, we have $\|G_n(t,u)^*x^*\|_H \leqslant C\|R^*x^*\|_H$, for a suitable constant C and $x^* \in E^*$ and, moreover, $G_n(t,u)^*x^* \to G(t,u)^*x^*$ in H.

Finally, let us address conditions (A1) – (A3). Let us first note that in this situation, the domains $\mathsf{D}(A_n)$ vary with n. However, if $1+\frac{1}{q} \geqslant 2\theta$, in particular if $0 \leqslant \theta < \frac{1}{2}$, then the complex interpolation space $[E, \mathsf{D}(A_n)]_\theta$ is (as a set) independent of n. However, to apply our results in this situation, we have to work on the fractional domain spaces (cf. the approximation results in the appendix which we use in the proof of our results) and we have to verify the estimates in (A3). By [**?**, Theorem 2.3] every operator A_n has a bounded H^∞-calculus, in particular, it has bounded imaginary powers. Therefore, see [**?**, Theorem 6.6.9], the fractional domain spaces are isometrically isomorphic to the complex interpolation spaces. Inspecting the proof of these results, the reader may check that if our assumptions on a_n, b_n and c_n are uniform in n, then the fractional domain spaces are isometrically isomorphic to $H^{2\theta,q}_{\{\mathscr{B}\}}(\mathscr{O})$ with constants independent of n, i.e. (A3) holds.

It remains to verify the strong resolvent convergence in (A2). This is most conveniently proved by rewriting our operators in divergence form. It is easy to see that for $q = 2$, the operator $-A$ is associated to the closed sectorial form

$$\mathfrak{a}[u,v] := \int_{\mathscr{O}} a(x)\nabla u(x)\overline{\nabla v(x)} + \tilde{b}(x)\nabla u(x)\overline{v(x)} + c(x)u(x)\overline{v(x)}\,dx$$

with domain $\mathsf{D}(\mathfrak{a}) = H^1(\mathscr{O})$, where the modified coefficients $\tilde{b}$ are given by $\tilde{b}_j = b_j - \sum_{i=1}^d \frac{\partial}{\partial x_i} a_{ij}$. Thus, under uniform boundedness and ellipticity assumptions, if $a_{ij}^{(n)} \to a_{ij}^{(\infty)}, D_i a_{ij}^{(n)} \to D_i a_{ij}^{(\infty)}, b_i^{(n)} \to b_i^{(\infty)}$ and $c^{(n)} \to c^{(\infty)}$, we obtain strong resolvent convergence of the operators for $q = 2$. This resolvent convergence extrapolates also to all $q \in [2,\infty)$. For proof of these facts, we refer to [**?**].

Appendix A. Convergence of analytic semigroups

In this appendix we prove some convergence results for analytic semigroups under assumptions (A1) – (A3). The lemmas A.1 and A.2 may be known to specialists, but since we could not find these results in the literature we include proofs for reasons of completeness.

The first lemma is used in the proof of Proposition 3.2.

Lemma A.1. *Assume (A1) – (A3). Then*

1. *For all $0 \leqslant \theta < \frac{1}{2}$ and $x \in E_\theta$ we have $S_n(t)x \to S_\infty(t)x$ in E_θ, uniformly on compact time intervals in $[0,\infty)$.*
2. *For all $0 \leqslant \theta, \kappa < \frac{1}{2}$ and $x \in E_{-\kappa}$ we have $A_nS_n(t)x \to A_\infty S_\infty(t)x$ in E_θ, uniformly on compact time intervals in $(0,\infty)$.*
3. *Let $\theta \in (0, \frac{1}{2})$ and $\lambda, \delta \geqslant 0$ satisfy $\lambda + \delta < \theta$. If $x_n \to x_\infty$ in E_θ, then $S_n(\cdot)x_n \to S_\infty(\cdot)x_\infty$ in $C^\lambda([0,T], E_\delta)$.*

Proof. For notational convenience, we will assume that $w < 0$ so that we may choose $w' = 0$ in the definition of the fractional domain spaces.

(1) Let $T > 0$ be given. For $x \in E_\theta$ we have

$$\begin{aligned}\|S_n(t)x - S_\infty(t)x\|_\theta &\simeq \|(-A_n)^\theta S_n(t)x - (-A_n)^\theta S_\infty(t)x\|_E \\ &\leqslant \|(-A_n)^\theta S_n(t)x - (-A_\infty)^\theta S_\infty(t)x\|_E \\ &\quad + \|(-A_\infty)^\theta S_\infty(t)x - (-A_n)^\theta S_\infty(t)x\|_E\,,\end{aligned} \tag{A.1}$$

where the implied constants in the first line may be chosen independently of n by (A3). Now observe that for $0 \leqslant t \leqslant T$,

$$\begin{aligned} \text{(A.2)} \quad & \|(-A_n)^\theta S_n(t)x - (-A_\infty)^\theta S_\infty(t)x\|_E \\ & \leqslant C_T \|(-A_n)^\theta x - (-A_\infty)^\theta x\|_E + \|S_n(t)(-A_\infty)^\theta x - S_\infty(t)(-A_\infty)^\theta x\|_E, \end{aligned}$$

where $C_T := \sup\{\|S_n(t)\|_{\mathscr{L}(E)} : 0 \leqslant t \leqslant T\,,\, n \in \mathbb{N}\} < \infty$ as a consequence of (A1). We note that the last term in (A.2) converges to 0 uniformly for $t \in [0, T]$ by (A1), (A2) and the Trotter-Kato theorem.

We now prove that, given a compact set $K \subseteq E_\theta$, we have $(-A_n)^\theta x \to (-A_\infty)^\theta x$ in E, uniformly for $x \in K$. This proves that also the first term on the right-hand side of (A.2) converges to 0, hence the first term on the right-hand side of (A.1) converges to 0. Moreover, since $\{S_\infty(t)x : 0 \leqslant t \leqslant T\}$ is compact in E_θ for all $x \in E_\theta$, it also follows that the second term on the right-hand side of (A.1) converges to 0, whence the proof of (1) is complete.

In view of the uniform boundedness of $(-A_n)^\theta$ as operators in $\mathscr{L}(E_\theta, E)$, to prove the convergence $(-A_n)^\theta x \to (-A_\infty)^\theta x$, uniformly on compact subsets of E_θ, it actually suffices to prove strong convergence on a dense subset of E_θ. To that end, pick $\eta \in (\theta, \frac{1}{2})$. Then E_η is a dense subset of E_θ, see [**?**, Proposition 3.1.1]. Moreover, for $x \in E_\eta$ we have $(-A_n)^\theta x = (-A_n)^{\theta-\eta}(-A_n)^\eta x$, hence, by [**?**, Corollary 3.3.6],

$$(-A_n)^\theta x = \frac{1}{\Gamma(\eta - \theta)} \int_0^\infty t^{\eta-\theta-1}(-A_n)^\eta S_n(t)x\,dt\,.$$

Now note that

$$\|t^{\eta-\theta-1}S_n(t)(-A_n)^\eta x\|_E \leqslant t^{\eta-\theta-1}Me^{wt} \sup_{n\in\mathbb{N}} \|(-A_n)^\eta\|_{\mathscr{L}(E_\eta,E)}\|x\|_\eta\,,$$

which is certainly integrable on $(0, \infty)$. Moreover, $(-A_n)^\eta S_n(t)x \to (-A_\infty)^\eta S_\infty(t)x$ for all $t \in (0, \infty)$ which, using (A1) and (A2), is easy to see by employing dominated convergence in a contour integral representation for $(-A_n)^\eta S_n(t)$.

Thus, by dominated convergence, $(-A_n)^\theta x$ converges in E to $(-A_\infty)^\theta x$, for all $x \in E_\eta$. This finishes the proof of (1).

(2) Fix $0 < \varepsilon < T$. We have

$$\begin{aligned} \|A_n S_n(t) - A_\infty S_\infty(t)x\|_\theta &\simeq \|(-A_n)^\theta A_n S_n(t)x - (-A_n)^\theta A_\infty S_\infty(t)x\|_E \\ &\leqslant \|(-A_\infty)^{\theta+1}S_\infty(t)x - (-A_n)^{\theta+1}S_n(t)x\| \\ &\quad + \|(-A_\infty)^\theta A_\infty S_\infty(t)x - (-A_n)^\theta A_\infty S_\infty(t)x\|_E\,. \end{aligned}$$

Convergence of the first term to 0, uniformly on $[\varepsilon, T]$, can be proved by a contour integral argument in the extrapolation space $E_{-\kappa}$. Convergence of the second term follows from the convergence of $(-A_n)^\theta x \to (-A_\infty)^\theta x$, uniformly on the set $\{A_\infty S_\infty(t)x : \varepsilon \leqslant t \leqslant T\}$, which is a compact subset of E_θ.

(3) Pick $\varepsilon > 0$ such that $\lambda + \delta + \varepsilon < \theta$. Then, for $t, s \in [0, T]$, we have

$$\begin{aligned} \|S_n(t)x_n - S_n(s)x_n\|_\delta &\simeq \|(-A_n)^\delta S_n(t)x_n - (-A_n)^\delta S_n(s)x_n\|_E \\ &\leqslant C(t-s)^{\lambda+\delta+\varepsilon}\|(-A_n)^{\lambda+\delta+\varepsilon}x_n\|_E \lesssim C(t-s)^{\lambda+\delta+\varepsilon}\|x_n\|_\theta\,, \end{aligned}$$

where C is a constant only depending on M and w in (A1). Furthermore, the implied constants in the first and the last step can be chosen independently of n. Since x_n is convergent, hence bounded, in C_θ, it follows that the sequence $(S_n(\cdot)x_n)_{n\in\mathbb{N}}$ is bounded in $C^{\lambda+\varepsilon}([0, T], E_\delta)$. Moreover, by (1), the continuity of the embedding $E_\theta \hookrightarrow E_\delta$ and the uniform boundedness of S_n on E_θ, it follows that $S_n(\cdot)x_n \to S_\infty(\cdot)x_\infty$ in $C([0, T]; E_\delta)$. This clearly yields that $S_n(\cdot)x \to S_\infty(\cdot)x_\infty$ in $C^\lambda([0, T], E_\delta)$. $\square$

If, in addition, (A4) holds, we have the following result.

Lemma A.2. *Assume (A1) – (A4). For all* $0 \leqslant \theta < \frac{1}{2}$ *and* $x \in B$ *we have* $S_n(\cdot)x \to S(\cdot)x$ *in* $C([0,T];B)$.

Proof. By Lemma A.1, we have $S_n(\cdot)x \to S(\cdot)x$ in $C([0,T];E_\theta) \hookrightarrow C([0,T];B)$ for all $x \in E_\theta$. By the density of E_θ in B and the uniform exponential boundedness of S_n^B, this extends to all $x \in B$. □

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